

Banking: A Mechanism Design Approach*

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Abstract

We study banking without a priori assumptions about what banks are, who they are, or what they do, using mechanism design. Given preferences, technologies, and certain frictions – including limited commitment and imperfect monitoring – we describe the set of incentive feasible allocations and interpret the outcomes in terms of institutions that resemble banks. Our bankers endogenously accept deposits, and their liabilities help in payments. This activity is essential in the sense that if it was ruled out the set of feasible allocations would be inferior. We discuss how to determine how many and which agents play the role of banks. Agents who are more connected to the market are better suited for this role since they have more to lose by renegeing on obligations.

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1 Introduction

The goal of this paper is to study banking without making a priori assumptions about what banks are, who they are, or what they do. To this end we adopt the approach of mechanism design. This method, in general, begins by describing an economic environment, including a set of agents, preferences and technologies, plus some details concerning what one might call frictions, regarding specifications of spatial or temporal separation, information problems, commitment issues, and so on. One then tries to describe the set of allocations that are attainable, respecting both resource and incentive feasibility constraints; sometimes one also derives properties of optimal allocations, given some optimality criterion. One then looks at what is feasible, or perhaps a subset, such as what is efficient, and tries to interpret the outcomes in terms of institutions that we observe in actual economies. We want to see if something that looks like banking emerges as an outcome of this exercise. To reiterate the idea, we do not take a bank as a primitive concept – our primitives are preferences, technologies, and certain frictions – because we want banking to be endogenous.

Recently mechanism design and its developers have been honored with a Nobel prize, and much has been written about the approach and its virtues.¹ From our perspective, which is more of a macro perspective, we want to mention some work of Townsend (1989). He asks if institutions we see in the world, such as certain more or less complete credit and insurance arrangements, can be derived from simple but internally consistent economic models, where by internal consistency we mean that one cannot simply assume a priori that some markets are missing, contracts are incomplete, prices are sticky, etc. Of course, something that looks like missing markets, incomplete contracts, etc., may emerge, but one ought to lay out the environment explicitly and derive this endogenously.² Very simple models, with minimal frictions,

¹See http://nobelprize.org/nobel_prizes/economics/laureates/2007/ecoadv07.pdf.

²As Townsend (1988) put it: “The competitive markets hypothesis has been viewed primarily as a postulate to help make the mapping from environments to outcomes more precise (...) In the end though it should be emphasised that market structure should be endogenous to the class of general

do not generate arrangements that resemble those in actual economies: those models predict credit or insurance arrangements work too well. So, one asks, what additional complications can be introduced to bring the models more in line with what we see? This is the method adopted here.

Obviously some frictions are needed in our exercise, since frictionless models, like Arrow-Debreu, do not have a role for banking. Similarly, as has been discussed often, frictionless models have no role for many institutions that facilitate the process of exchange, including money. A long-standing challenge in monetary economics is to ask what frictions make money essential, in the following precise sense: money is essential when the set of allocations that satisfy incentive and other feasibility conditions is bigger or better with money than without it (Wallace 2001, 2008). One does not take money as a primitive in this analysis. We ask a similar question: when are banks essential in the same sense. In our environment, it is a good idea for the planner – or the mechanism – to have some agents, determined endogenously, perform certain functions that resemble salient elements of banking. These functions can be described as having these agents accept deposits, and having their liabilities, or claims on those deposits, used for payments. This activity is essential in the sense that if it was ruled out the set of feasible allocations would be inferior.

There is a large and interesting literature on banking, surveyed e.g. by Gorton and Winton (2002) and Freixas and Rochet (2008), but it is most different from our approach. Most of this literature is based on the idea that informational asymmetries – whether ex-ante (adverse selection), interim (moral hazard) or ex-post (costly state verification) – is detrimental to the efficient channeling of funds from investors to entrepreneurs. For instance, investors are unable to perfectly diversify their risk and invest less than the efficient amount. In this large literature we may distinguish among three main interpretations of banks. Work originating from Diamond and

equilibrium models at hand. That is, the theory should explain why markets sometimes exist and sometimes do not, so that economic organisation falls out in the solution to the mechanism design problem.”

Dybvig (1983) interprets banks as liquidity pools or coalitions of agents providing insurance against liquidity shocks.³ A second approach pioneered by Leland and Pyle (1977) and developed by Boyd and Prescott (1986) interprets bank as information sharing coalitions that can induce agents to truthfully reveal the quality of the entrepreneur's endeavor. A third approach, following Diamond (1984), interprets banks as "delegated monitors". According to this view if monitoring is subject to economies of scale, investors are too small to individually finance a project and the cost of delegation is also small, it is optimal to delegate monitoring to an agent that acts like a bank.

These papers provide insightful analyses, but sometime take as given many institutional details. Diamond Dybvig, for example, do not interpret the bank as an agent in the model, but as a planner or a contract, and do not derive endogenously which agents will be banker(s). In the papers that emphasize information sharing or delegated monitoring, banks are agents, but their role is restricted to solving the informational problems arising from the financing of investment projects. The fact that the liabilities of banks are crucial in the payment system is not at all considered.⁴

Historically, it can be argued that modern banking starts with the English goldsmiths, whose role originally was to hold deposits for safekeeping against the possibility of theft or other loss, but soon receipts for these deposits began to circulate as a means of payment in the form of banknotes and checks (see the references in He at al. 2005 for extended discussions). There is virtually nothing in the literature emanating from Diamond-Dybvig (1983) or Diamond (1984) on these issues, which are not only of historical but also contemporary relevance, given modern developments concerning private monies, the design of alternative payment systems, etc. (a recent exception is

³The literature on the Diamond and Dybvig model is very large. Among others we can quote Jacklin (1987), Wallace (1988) and (1990), Peck and Shell (2003), Green and Lin (2003), Andolfatto, Nosal and Wallace (2007) and Ennis and Keister (2008).

⁴A recent model where banks play a role in the payment system and, at the same time, face a private information friction as in Townsend (1979), Diamond (1984) and Gale and Hellwig (1985) is the one proposed by Andolfatto and Nosal (2008).

Huangfun and Sun 2008). While there have been some papers on the role of commercial banks in the payment process (Kiyotaki and Moore 2006, Andolfatto and Nosal 2008, Williamson and Sanches 2008, He et. al. 2005 and the references therein), they typically do not take the mechanism design approach, and certainly do not ask who should be a banker. And while some papers do use mechanism design, they tend to focus on issues other than those studied here (see e.g. Cavalcanti and Wallace 1999a, 1999b, Mills 2008). So, while much in the existing research on banking is useful, we think there is room for more.

We present a simple environment where there are gains from trade that would be relatively easy to exploit, except for the fact that agents cannot perfectly commit. We highlight limited commitment because to us banking concerns intertemporal allocation issues that revolve around some notion of credit, which leads us to take seriously agents' incentives to make good on their obligations. We allow agents to use stored output as collateral to ameliorate commitment problems, but emphasize that this may not work so well when the collateral can be easily liquidated, which makes debtors' promises to deliver future payments out of stored goods no more credible than promises to deliver out of future production.⁵ One implication of this observation is that a potentially useful device may be delegated storage: if you deposit your output with a third party, who has either a lower ability or incentive to liquidate these deposits for strategic reasons, others may be more willing to offer you credit, which can be interpreted as using your claims on deposits to help back certain promises, or more generally help to facilitate transactions. This to us resembles banking.

Stepping outside the formal model for a moment, the idea is that sellers may accept payment from a buyer in the form of an obligation of a third party – which may take the form of a note, check, credit or debit card, or other instrument issued by a commercial bank – when the same sellers would not accept the buyer's personal IOU. One may, however, ask why some parties have a lower ability or incentive to

⁵This is related to the idea in Kiyotaki and Moore (2008) that borrowers can commit to deliver some but not all of their collateral.

renege on obligations. In our approach, future rewards and punishments are used to mitigate strategic behavior, but monitoring is imperfect: strategic deviations from desired behavior are detected only probabilistically. Agents with a higher likelihood of being monitored have a greater incentive to make good on obligations, and hence are better suited to take on the responsibility of banking and hold deposits. This is not a new insight – e.g. a higher likelihood of being monitored is the key characteristic underlying banks in the Calvo-Wallace model – but it seems valid. However, it raises the question of why some agents are more easily monitored, especially if one thinks monitoring ought to be endogenous.

To address this, we assume that agents have different probabilities of gaining from market activities in a given period – i.e. they have different stakes in the economic system. Even with equal monitoring probabilities, those with a higher stake in the system are less inclined to deviate from proscribed behavior. Consistent with experience, individuals with a greater connection to the market are better suited to play the role of bankers, since they have more to lose by renegeing on obligations. We then ask which agents should be endogenously monitored, and how much should they be monitored, when it is costly. It is desirable to select agents with a greater connection to the market to be bankers, and to spend more on monitoring them. One can also ask how many bankers are desirable. The answer hinges on the following tradeoff: having fewer bankers entails lower monitoring costs, but leads to more deposits per bank, which increases incentives to misbehave. All of this comes directly out of our mechanism design approach, without a priori assumptions about what is a bank, who is a bank, or what do banks do. Hence, we think our approach provides some new insights into banking theory.

2 The Environment

Time is discrete and continues forever. There are $N \geq 1$ different sectors, or perhaps islands, that in general are heterogeneous, but for now we consider a representative

sector. This consists of a set \mathcal{A} of ex-ante homogenous agents, and a measure defined on its set of measurable subsets of with range $[0, 1]$, such that each period we partition \mathcal{A} into three groups, \mathcal{A}_0 , \mathcal{A}_1 and \mathcal{A}_2 with measures γ_0, γ_1 and γ_2 . Agents in group \mathcal{A}_0 are called nontraders: they do not consume or produce and derive a payoff normalized to 0 that period. Agents in groups \mathcal{A}_1 and \mathcal{A}_2 are called traders of type 1 and 2: they potentially produce, consume, and derive payoffs described below. Each period, agents take as given they will belong to group \mathcal{A}_i with probability γ_i , $i = 0, 1, 2$ (thus agent types are not fixed but change over time). This setup captures the idea that different agents can have a different stake in the economy – a bigger γ_0 e.g. means that agents are less likely to participate in market production and consumption in a given period. Because agents here cannot perfectly commit to future actions, dynamic incentive conditions will play a critical role, and these will depend in interesting ways on agents' connection to the market as represented by the γ_i 's.

We now describe traders in a given period. There are two goods, 1 and 2. Agents in \mathcal{A}_1 consume good 1 and produce good 2, while agents in \mathcal{A}_2 consume good 2 and produce good 1. Letting x_i and y_i denote consumption and production by an agent of type i in a given sector, utility $U^i(x_i, y_i)$ is increasing in x_i , decreasing in y_i , satisfies the usual convexity conditions, and $U^i(0, 0) = 0$. A key friction is temporal separation: we divide each period in two and assume good i must be consumed in subperiod i . This generates a role for credit, since type 1 consumes before type 2. To have some notion of collateral, we assume good 2 is produced in the first subperiod and stored to the second by type 1 (only the producer of good 2 can store it). A unit of good 2 stored in the first subperiod returns $1 + \rho$ units in the second; for now, set $\rho = 0$.⁶ Although goods can be stored across subperiods, they fully depreciate across periods. To make collateral less than perfect, assume that if type 1 consumes x_2 units good 2 out of storage, he gets liquidation utility λx_2 . Note that type 1 gets liquidation utility from any good 2, even if it was produced by another type 1 agent, including one from a different sector. However, only goods produced within a sector

⁶We consider the case where $\rho > 0$ later on.

enter $U^i(x_i, y_i)$. Also, assume $U^1(x_1, y_1) + \lambda y_1 \leq U^1(x_1, 0)$ for any x_1 , so there is no point to type 1 producing for their own consumption.⁷

We focus on symmetric and stationary allocations. Such an allocation is given by a vector of consumption and production in each sector (x_1, y_1, x_2, y_2) , plus, as we discuss later, descriptions of cross-sector transfers, storage and liquidation. A planner collects all production and allocates it to consumers.⁸ An allocation is resource feasible, assuming for now no transfers across sectors or liquidation, if $\gamma_2 y_2 = \gamma_1 x_1$ and $\gamma_1 y_1 = \gamma_2 x_2$, and hence can be summarized by (x_1, x_2) . To reduce notation without affecting the main results, we set $\gamma_1 = \gamma_2 = \gamma$ and $\gamma_0 = 1 - 2\gamma$. Finally, it is convenient to let $\theta = \pi\gamma\beta/(1 - \beta)$, where β is the discount factor across periods, and π represents a monitoring technology, so that a deviation by an agent from any recommendation of the planner is detected with probability π . When a deviation is observed, the agent is punished with autarchy – banishment from future market production and consumption. Notice θ measures the risk of deviating: the probability of being detected π , times one's connection to the market γ , times the weight one puts on future participation $\beta/(1 - \beta)$.

Some special remarks are worth mentioning. First, U^i may be additively separable, in which case we can assume it is quasi-linear without loss in generality by an appropriate choice of units, say $U^i = u^i(x_i) - x_j$. Notice this says U^i is linear in one good and U^2 in the other; it would be different to make them both linear in the same

⁷A few details concerning the specification deserve comment. First, the linearity of the liquidation value λx_2 is only to ease the presentation; we could use e.g. $U^1(x_1, y_1, x_2)$, but this adds little other than notation. Also, we do not allow goods produced in different sectors to enter $U^i(x_i, y_i)$ because this means any transfers across sectors will be due only to incentive considerations, not the usual reasons for trade. Similarly, we can have a return $\rho \neq 1$, but $\rho = 1$ implies storage occurs only for incentive reasons.

⁸Thus, traders deal directly with the planner, or mechanism, which is important only to the extent that we do not want to necessarily restrict them to bilateral trade. Note that the planner does not store good 2: it must be stored by its producer type 1 and then delivered to the planner in subperiod 2, who then gives it to type 2. Obviously we do not let type 2 agents store good 2 for themselves, as this eliminates entirely temporal separation and our discussion of credit.

good, say $U^1 = x_1 - v(x_2)$ and $U^2 = u(x_2) - x_1$, meaning all the gains from trade accrue from x_2 , although x_1 may still be relevant for incentive reasons. As an extreme version, we can eliminate x_1 altogether, and given this, we can again assume without loss in generality x_2 enters one function linearly, say $U^1 = -x_2$ and $U^2 = u(x_2)$. In this case there is nothing one can do to reward type 2 for producing within a period, so incentives revolve only around future promises, whereby θ becomes all the more relevant. More generally, we can reward type 2 intratemporally and intertemporally. In the separable case, production and consumption are sunk when type 2 considers delivering versus liquidating storage, so the only relevant promises involve the future, but this is not true when U^i is not separable. For this reason, and since it nests the other cases, we use the general specification $U^i(x_i, x_j)$.⁹

3 A One-Sector Model

We are interested in the set of incentive feasible, or IF, allocations that are stationary and symmetric. Thus, the mechanism, or the planner, proposes a stationary and symmetric recommendation for production and consumption, summarized by (x_1, x_2) , given resource feasibility. Any such recommendation can be implemented provided it is an IF allocation. In the baseline model agents cannot commit to future actions and hence are allowed to deviate from a recommendation whenever they like, albeit at the risk of punishment. However, we begin by describing for benchmark purposes what happens when agents can commit. Also, in this section we focus on $N = 1$ sector, so that we can ignore for now any reallocation of goods across sectors.

⁹The separable and quasi-linear cases are not necessary for much other than reducing notation and algebra in particular applications; we do not use such assumptions the way they are used in monetary models along the lines of Lagos and Wright (2005) e.g., despite a superficial resemblance in the environments, with each period split in two subperiods, something that looks like a dynamic double-coincidence problem, etc.

3.1 Benchmark Allocations

As we said, in the baseline model agents cannot commit, but as a benchmark we consider what happens when they can. In this case, the planner faces only the ex ante participation constraint

$$S(x_1, x_2) \geq 0, \quad (1)$$

where $S(x_1, x_2) \equiv U^1(x_1, x_2) + U^2(x_1, x_2)$ is defined as the total surplus available per period. Intuitively, with full commitment, IF allocations only have to generate more total surplus than autarchy. In particular, in any given period we can have $U^i < 0$ for one i , as it must be e.g. in the special case $U^1 = -x_2$ and $U^2 = u(x_2)$, as long as the sum S is positive. Although we want to characterize the entire set of IF allocations, consider the ex ante Pareto optimal allocation (x_1^*, x_2^*) (later we consider optimality at various points in time within a period). Assuming an interior solution, which obtains under standard conditions, this solves $\partial U^i / \partial x_i + \partial U^j / \partial x_i = 0$ for $i = 1, 2$. In any case, under full commitment, (x_1^*, x_2^*) is always IF.¹⁰

In this stationary environment, full commitment is obviously equivalent to being able to commit within a period, before types are realized, even if agents cannot commit across periods. Consider as another benchmark what happens when agents can only commit within a period after types are realized – which is equivalent to eliminating temporal separation and assuming goods 1 and 2 are consumed simultaneously. We now have to ensure both types of traders want to participate each period:

$$U^i(x_i, x_j) + \frac{\gamma\beta}{1-\beta} S(x_1, x_2) \geq \frac{(1-\pi)\gamma\beta}{1-\beta} S(x_1, x_2), \quad i = 1, 2, \quad j \neq i.$$

The left side gives the payoff to type i who follows the recommendation (x_1, x_2) ; the right side gives the payoff to a deviator, who gets no production or consumption that period, and with probability π is punished with autarchy but with probability $1 - \pi$ gets away with it. Notice that a deviation here means neither producing

¹⁰In the case where $U^1 = u^1(x_1) - x_2$ and $U^2 = u^2(x_2) - x_1$, e.g., the ex ante Pareto optimal allocation solves $\partial u^i / \partial x_i = 1$, $i = 1, 2$; and in the case where $U^1 = x_1 - v(x_2)$ and $U^2 = u(x_2) - x_1$, x_2 solves $\partial u / \partial x_2 = \partial v / \partial x_2$ while x_1 is irrelevant.

nor consuming that period; alternatives, like assuming it means consuming but not producing (which should be interpreted as a change in the timing) can be analyzed similarly.

Using $\theta = \pi\gamma\beta/(1 - \beta)$, the above inequality reduces to the following dynamic participation constraint:

$$U^i(x_i, x_j) + \theta S(x_1, x_2) \geq 0, \quad i = 1, 2, \quad j \neq i. \quad (2)$$

Notice (2) implies (1) but not vice versa. Recall that with full commitment we can always have $U^i < 0$ as long as $S \geq 0$. Now it is possible to have $U^i < 0$ only if promises of future rewards are sufficiently great, and this depends on θ . Therefore, with imperfect commitment we may have to use future rewards to provide incentives. In particular, we may not be able to support the ex ante Pareto optimal allocation here. Consider e.g. the case $U^1 = -x_2$ and $U^2 = u(x_2)$, where x_2^* solves $\partial u/\partial x_2 = 1$ and x_1 is of course irrelevant. Now IF requires (2) for type 1, which reduces to $-x_2^* + \theta [u(x_2^*) - x_2^*] \geq 0$. This is satisfied iff θ is large.

As a final benchmark, consider what happens when x_2 is actually produced in the second subperiod. Now to get type 1 to produce good 2 after having consumed good 1, we need

$$U^1(x_1, x_2) + \frac{\gamma\beta}{1 - \beta} S(x_1, x_2) \geq U^1(x_1, 0) + \frac{(1 - \pi)\gamma\beta}{1 - \beta} S(x_1, x_2),$$

which reduces to

$$U^1(x_1, x_2) - U^1(x_1, 0) + \theta S(x_1, x_2) \geq 0. \quad (3)$$

This implies (2) for type 1. Hence, IF allocations satisfy (2) for type 2 and (3) for type 1. This model captures a notion of within-period credit: type 2 produce and type 1 consume good 1 in the first subperiod, in exchange for promises concerning the second subperiod, as well as future periods, and (3) can be interpreted as a repayment constraint saying type 1 has the incentive to honor obligations. While this is interesting, we use a slightly different version as our baseline model of credit which introduces collateral considerations.

3.2 Baseline Model

In our baseline model, type 1 produce good 2 in the first and store it to the second subperiod, instead of producing x_2 in the second subperiod as in the previous benchmark. Recall type 1 derives liquidation value λx_2 if he consumes good 2 out of storage, leading to the following repayment constraint:¹¹

$$-\lambda x_2 + \theta S(x_1, x_2) \geq 0. \quad (4)$$

If $\lambda = 0$, (4) does not bind, as it follows from participation condition (1), and IF allocations are defined only by (2). Intuitively, $\lambda = 0$ implies, when it comes time for type 1 to deliver x_2 , the production cost is sunk. In this case collateral works well. If λ is big, however, either (4) or (2) may bind for type 1. Generally, the set of IF allocations is given by (2) for both types and (4) for type 1.

In terms of the economics, a liquidation option implies an opportunity cost to delivering the goods, making x_2 a poor collateral: a promise to deliver it in the future is not necessarily more credible that a promise to produce x_2 in the future. Given λ , collateralized credit, like other forms of credit, only works if θ is big, which means agents are patient (β big), have a sizable connection to future markets (γ is big), and are easily monitored (π is big). When $U^1 = u^1(x_1) - x_2$ is separable the repayment condition (4) reduces to $-\lambda x_2 + \theta S(x_1, x_2) \geq 0$ while the dynamic participation condition (2) reduces to $u^1(x_1) - x_2 + \theta S(x_1, x_2) \geq 0$, and the former is more stringent iff type 1's surplus $u^1(x_1) - x_2$ in the period exceeds the opportunity

¹¹In case it is not obvious, this comes from simplifying

$$U^1(x_1, x_2) + \frac{\gamma\beta}{1-\beta} S(x_1, x_2) \geq U^1(x_1, x_2) + \lambda x_2 + \frac{(1-\pi)\gamma\beta}{1-\beta} S(x_1, x_2).$$

The payoff to a deviator on the right involves consuming x_1 and producing x_2 in the first and liquidating x_2 in the second subperiod, whence with probability π the agent is punished and with probability $1 - \pi$ gets away with it. When $U^1(x_1, 0) - U^1(x_1, x_2) = \lambda x_2$, which occurs when $U^1(x_1, x_2) = u(x_1) - \lambda x_2$, notice that (4) is equivalent to (3) from the benchmark model where good 2 is produced in the second subperiod, rather than stored. But while the algebra looks the same, storage allows us to discuss collateral, and later banks, more naturally.

cost of the liquidation $-\lambda x_2$. This is particularly transparent when $U^1 = -x_2$, in which case the repayment condition is more stringent iff the marginal opportunity cost of liquidation λ exceeds the marginal production cost that has been normalized to 1.

Figure 1 presents two examples, a symmetric one with $U^1 = \sqrt{x_1} - x_2$ and $U^2 = \sqrt{x_2} - x_1$ on the left; and an asymmetric one $U^1 = x_1 - x_2$ and $U^2 = \sqrt{x_2} - x_1$ on the right. The IF set for the baseline model is shown in dark blue, the case of full commitment in light blue, and the case of commitment within a period after types are realized in medium blue. Also shown is the set of Pareto optimal allocations using two different criteria: payoffs of types 1 and 2 at the start of a period after types have been realized but before production takes place in red; and payoffs in the middle of a period after production has taken place but before type 1 delivers x_2 in black. The ex ante Pareto optimal allocation before types are realized is (x_1^*, x_2^*) , as discussed above, which occurs at the intersection of the black and red curves. A detailed derivation of the different optimal allocations is contained in the Appendix. Here we do not dwell too much on these outcomes, since different welfare criterion, including ones other than Pareto criteria, may be relevant. For now we study the entire IF set. But it may be worth mentioning that the Pareto optimal allocations are not in this set when θ is small, as seen in the right pannel.

4 A Multi-Sector Model

We now analyze the model with $N > 1$ sectors. Since it will suffice for most of the interesting points, we focus on $N = 2$, and label the two sectors a and b . These sectors may be heterogeneous: in sector $i = a, b$, each period with probability γ^i agents are type 1^i or 2^i traders, producing and consuming goods 1^i and 2^i as above, and are nontraders with probability $1 - \gamma^i$. Also, we can detect deviations with probability π^i , and type 1^i agents have liquidation value λ^i in sector i . We allow type 1^i to

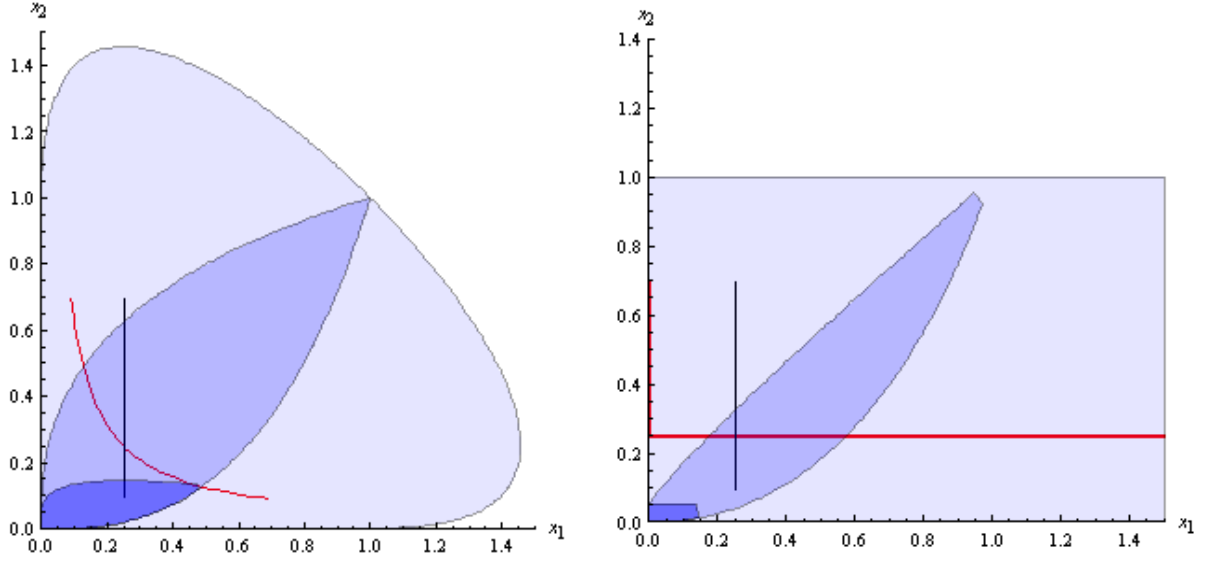


Figure 1: IF and PO allocations in two examples.

store and liquidate either good 2^i produced in sector i or good 2^j produced in sector $j \neq i$. There are no gains from trade across sectors for pure mercantile reasons, since the goods produced in sector i do not enter the utility function U^j and vice-versa. However, suppose t units of good 2 is produced by type 1^b in sector b and transferred to type 1^a in the other sector, and the latter liquidate it for consumption. Assuming no one liquidates any of their own production, which is never efficient anyway since $U^1(x_1^i, y_1^i) + \lambda^i y_1^i \leq U^1(x_1^i, 0)$, utility for type 1^b is $U^1(x_1^b, x_2^b + t)$, since they produce x_2^b for type 2^b in their own sector and t for type 1^b in the other sector, while utility for type 1^a is $U^1(x_1^a, x_2^a) + \lambda^a \gamma^b t / \gamma^a$ since the total transfer is $\gamma^b t$ and is redistributed among a measure γ^a of agents.

We are not especially interested in direct transfers, since they yield no direct gains from trade, and in fact we think of it as yielding a reduction in total surplus. Still, direct transfers can have interesting incentive effects, and we need to analyze this for the following reason. What we are more interested in is the case where d units of good 2 is produced by type 1^b and transferred to type 1^a , and the latter do not liquidate it but store d and transfer it back to type 2^b agents in sector b . We are

interested in this delegated storage because, we will argue, it resembles banking. We want to show that delegated storage is essential in the sense that it can change the set of IF allocations for the better. Now, once we allow delegated storage d , a direct transfer t is also feasible since we allow type 1 agents to liquidate goods produced in the other sector, and direct transfers also change the set of IF allocations. We will argue, however, that delegated storage does more than direct transfers, and to make this case we need to first analyze what transfers do.

Before we begin, a few details deserve mention. First, we emphasize that although one can think of t in the above discussion as a lumpsum tax, an agent of type 1^b does not have to pay this tax. The planner can only recommend that type 1^b agents voluntarily contribute t , which they will agree to iff it satisfies the relevant incentive conditions. Second, without loss of generality, we often focus on the case where γ^i , λ^i and π^i are such that $\theta^b \geq \theta^a$. This means type 1^b have, other things equal, less of a problem with commitment issues than 1^a since they weight more heavily future rewards against short-term opportunism. Finally, we only consider a transfer t in one directions, say from sector b to sector a , without loss of generality, since transfers in the other direction are symmetric and it is never useful to have simultaneous transfers in both directions.

4.1 Direct Transfers

The surpluses in sectors a and b are now

$$\begin{aligned} S^a(x_1^a, x_2^a, t) &\equiv U^1(x_1^a, x_2^a) + U^2(x_2^a, x_1^a) + \lambda^a \gamma^b t / \gamma^a \\ S^b(x_1^b, x_2^b, t) &\equiv U^1(x_1^b, x_2^b + t) + U^2(x_2^b, x_1^b). \end{aligned}$$

Obviously we need $S^i(x_1^i, x_2^i, t) \geq 0$, but this is not binding, given the dynamic participation conditions. For type 2^i these are

$$U^2(x_2^i, x_1^i) + \theta^i S^i(x_1^i, x_2^i, t) \geq 0, \quad i = a, b, \quad (5)$$

and for 1^i they are

$$U^1(x_1^a, x_2^a) + \lambda^a \gamma^b t / \gamma^a + \theta^a S^a(x_1^a, x_2^a, t) \geq 0 \quad (6)$$

$$U^1(x_1^b, x_2^b + t) + \theta^b S^b(x_1^b, x_2^b, t) \geq 0. \quad (7)$$

Finally, the repayment constraints for types 1^i are

$$-\lambda^i x_2^i + \theta^i S^i(x_1^i, x_2^i, t) \geq 0, \quad i = a, b. \quad (8)$$

The set of IF allocations for a given t satisfies (5)-(8). If $t = 0$ these constraints reduce to (2) for both types and (4) for type 1^i in each sector, as in the previous section.

The key observation is that we can use $t > 0$ to relax the constraints in sector a , albeit at the cost of tightening them in sector b . Notice that $t > 0$ affects the repayment constraint (8) only through the future surplus $S^i(x_1^i, x_2^i, t)$, although it affects the participation conditions (6) and (7) directly. A transfer in the other direction of course has the opposite effect. Thus, $t \neq 0$ can be used to alter the set of IF allocations, and whenever the constraints are binding in one sector but not the other, with $t \neq 0$ we can expand the set of IF allocations. Therefore one would say that transfers are essential in the technical sense used in the Introduction – although we do not want to make a big deal out of this and are not suggesting it is surprising. Still, we want to see just how much we can accomplish with these simple transfers between types 1^a and 1^b . To this end, we begin by asking, what is the maximum feasible transfer t from sector b to sector a ? To answer this, we ignore for the present sector a , and choose the allocation (x_1^b, x_2^b) that allows us to extract the biggest t , subject to (5)-(8). This is a standard maximization problem, and there exists a unique solution, say $(\hat{x}_1^b, \hat{x}_2^b, \hat{t})$.

Since the left sides of the constraints are increasing in θ^b , so is \hat{t} : when agents are more patient, more connected to the market, or more frequently monitored, we can extract more resources from them. Some examples may illustrate how this works. First, suppose $U^1(x_1, x_2) = x_1 - x_2$ and $U^2(x_2, x_1) = u(x_2) - x_1$. In this economy all the ex ante gains from trade come from x_2 , while x_1 is simply a transfer from type 2

to type 1. Further assume $\lambda^b = 0$. Then IF allocations in sector b solve

$$u(x_2^b) - x_1^b + \theta^b [u(x_2^b) - x_2^b - t] \geq 0 \quad (9)$$

$$x_1^b - u(x_2^b) + (1 + \theta^b) [u(x_2^b) - x_2^b - t] \geq 0 \quad (10)$$

$$u(x_2^b) - x_2^b - t \geq 0. \quad (11)$$

We achieve the maximum transfer at $\hat{x}_2^b = x_2^*$, $\hat{x}_1^b = u(x_2^*)$, and $\hat{t} = u(x_2^*) - x_2^*$, where in this case x_2^* solves $u'(x_2) = 1$. Notice $S^b(\hat{x}_1^b, \hat{x}_2^b, \hat{t}) = 0$, and the constraints (9)-(11) all hold with equality. Thus, we can get the agents to produce the x_2^* that maximizes the surplus, and then tax away the entire surplus with \hat{t} , and because $\lambda^b = 0$ we do not have to worry about repayment by type 1. In this extreme case θ^b is actually irrelevant since $S^b(\hat{x}_1^b, \hat{x}_2^b, \hat{t}) = 0$.

As another example, let $U^1(x_1, x_2) = -x_2$ and $U^2(x_2, x_1) = u(x_2)$, so that again all the ex ante gains from trade come from x_2 but now x_1 cannot be used to transfer utility from type 2 to type 1. In this example set $\lambda^b = 1$. Now the IF allocations solve

$$u(x_2^b) + \theta^b [u(x_2^b) - x_2^b - t] \geq 0 \quad (12)$$

$$-x_2^b - t + \theta^b [u(x_2^b) - x_2^b - t] \geq 0 \quad (13)$$

$$-x_2^b + \theta^b [u(x_2^b) - x_2^b - t] \geq 0 \quad (14)$$

Clearly (12) and (14) are not binding, given $t \geq 0$, so we need only worry about the dynamic participation constraint (13). The maximum amount we can tax this economy is

$$\hat{t} = \frac{\theta^b}{1 + \theta^b} u(\hat{x}_2^b) - \hat{x}_2^b \quad (15)$$

where $u'(\hat{x}_2^b) = (1 + \theta^b) / \theta^b$. Notice $\hat{x}_2^b < x_2^*$. Also notice $\partial \hat{x}_2^b / \partial \theta^b > 0$ and $\partial \hat{t} / \partial \theta^b > 0$. In this economy we have to provide type 1 the incentive $\theta^b [u(x_2^b) - x_2^b - t]$ to cover the cost of producing for type 2 consumption plus the transfer to the other sector, $x_2^b + t$. At the time when he is supposed to deliver x_2 out of storage we have to provide type 1 only with the incentive to cover the opportunity cost of liquidation $\lambda^b x_2^b = x_2^b$, which is therefore not binding when $t \geq 0$.

In either of these examples, as in the general case, the key point is for some allocations we may want to implement in sector b we can extract $t > 0$ and use it to subsidize type 1^a agents in the other sector, thereby relaxing constraints and expanding the set of IF allocations. Again, it is no surprise taxes and transfers can increase the set of IF allocations; we only present these results to show that delegated storage, which resembles banking, can do more than simple taxes and transfers.

4.2 Delegated Storage: Intermediation

Suppose now the planner collects production and redistribute goods for consumption, and/or storage in both sectors. The difference is that, in addition to transfers, the planner now deposits $d \geq 0$ units of consumption good of type 1^a with type 1^b .¹² Using storage, the planner still faces the dynamic participation constraints (5) for type 2^i and (6)-(7) for type 1^a and 1^b respectively. With deposits d , type 1^a only now store $x_2^a - d$ units. Therefore, their repayment constraint becomes

$$-\lambda^a (x_2^a - d) + \theta^a S^a(x_1^a, x_2^a, t) \geq 0 \quad (16)$$

Similarly, type 1^b now store $x_2^b + \gamma^a d / \gamma^b$ units, so that their repayment constraint is¹³

$$-\lambda^b (x_2^b + \gamma^a d / \gamma^b) + \theta^b S (x_1^b, x_2^b, t) \geq 0 \quad (17)$$

If $d > 0$ then (17) implies (8) in sector b while (8) in sector a implies (16). In addition, it is worth noticing that deposits only interact with transfers in the liquidation value and not in the continuation payoff. Finally, the planner faces the resource constraint¹⁴

$$0 \leq d \leq x_2^a \quad (18)$$

¹²It is obvious to extend the framework to allow agents in sector b to deposit goods with agents in sector a . It is easy to show that only $d \geq 0$ will maximize the set of incentive feasible allocations in sector a .

¹³Again, resource constraint implies that aggregate resources available for deposits are $\gamma^a d$, and therefore each agent 1^b get $\gamma^a d / \gamma^b$ in deposits.

¹⁴This resource constraint is $-x_2^b \leq d \leq x_2^a$, if we allow deposits of sector b in sector a .

With delegated storage, the set of feasible allocation \mathcal{F}_d is defined as those allocations $(x_1^a, x_2^a, x_1^b, x_2^b)$ with transfer t and deposits d such that (5)-(7) and (16)-(18) hold. We can now state our main result.

Theorem 1 *Deposits are essential, in the sense that for any feasible $d > 0$, $\mathcal{F}_0 \subseteq \mathcal{F}_d$ and for some parameters $\mathcal{F}_d \setminus \mathcal{F}_0 \neq \emptyset$.*

In words, intermediation is essential in the sense that some allocations are feasible with intermediation but are not feasible without.

Proof. First notice that any feasible allocations without deposits are also feasible when deposits are allowed. Indeed, it is enough for the planner to set $d = 0$. To show that more allocations are feasible, we use an example. Suppose that $\lambda^b = 0$. Then we show that there are some allocations in sector a that are only feasible when deposits are used. In particular, we set $t = \hat{t}$, to maximize the (net) transfer from sector b to sector a . Given $(x_1^b, x_2^b, t) = (\hat{x}_1^b, \hat{x}_2^b, \hat{t})$, all incentive constraints are satisfied in sector b . The planner then only has to consider incentive constraint in sector a . The relevant conditions (5), (6) and (16) become

$$\begin{aligned} U^2(x_2^a, x_1^a) + \theta^a S^a(x_1^a, x_2^a, t) &\geq 0 \\ -U^2(x_2^a, x_1^a) + (1 + \theta^a) S^a(x_1^a, x_2^a, t) &\geq 0 \\ -\lambda^a(x_2^a - d) + \theta^a S^a(x_1^a, x_2^a, t) &\geq 0 \end{aligned}$$

Therefore, for any allocation such that

$$\begin{aligned} \lambda^a x_2^a &\geq -U^2(x_2^a, x_1^a), \text{ and} \\ \lambda^a x_2^a &\geq U^2(x_2^a, x_1^a) \frac{\theta^a}{(1 + \theta^a)} \end{aligned}$$

increasing deposits will relax the repayment constraint and therefore expand the set of feasible allocations. As a matter of fact, set $d = x_2^a$, then the repayment constraint becomes $S^a(x_1^a, x_2^a, t) \geq 0$, which is the IR constraint. Hence the repayment constraint becomes redundant. ■

Notice that the proof relies on the fact that $\lambda^b = 0$ for all x . It is then feasible to store all the production of type 1^a with type 1^b without modifying their incentives. Indeed, when agents 1^b do not value liquidated good, they are perfectly able to commit instead of agents 1^a .¹⁵

Let us again illustrate how deposits are essential. First, suppose as in the previous section that $U^1(x_1, x_2) = x_1 - x_2$, $U^2(x_2, x_1) = u(x_2) - x_1$, $\gamma^a = \gamma^b$ and $\lambda^a > \lambda^b = 0$. With transfers t and deposits d , IF allocations in sector a solve

$$u(x_2^a) - x_1^a + \theta^a [u(x_2^a) - x_2^a + \lambda^a t] \geq 0 \quad (19)$$

$$x_1^a - u(x_2^a) + (1 + \theta^a) [u(x_2^a) - x_2^a + \lambda^a t] \geq 0 \quad (20)$$

$$-\lambda^a (x_2^a - d) + \theta^a [u(x_2^a) - x_2^a + \lambda^a t] \geq 0 \quad (21)$$

In the previous section, we know that $\hat{t} = u(x_2^*) - x_2^*$ is the maximum feasible transfer. Increasing t from zero to \hat{t} , relaxes all constraints. Also, increasing d from 0 relaxes the repayment constraint (21). Obviously, the repayment constraint is independent of x_1^a and it therefore defines an upper bound \bar{x}_2^a above which x_2^a is not feasible.¹⁶ Increasing d makes this upper bound bigger and therefore expands the set of feasible allocations. The graph below shows each set of feasible allocations for the case where 1) $t = 0$ and $d = 0$ in red, 2) $t = \hat{t}$ and $d = 0$ in green, and 3) $t = \hat{t}$ and $d = x_2^a$ in green and blue.

Clearly, as agents in sector b have no gain from liquidation, they are able to make good on their promise to deliver, even if they store a large amount of goods. The planner can exploit this commitment ability to fully relax the repayment constraint in sector a . Some readers may not be entirely convinced by this example, as it relies on assuming different preferences across the two sectors. This doubt should however be resolved with our next example.

¹⁵The planner could require that type 1^b deposits d^b with 1^a while they deposit d^a with type 1^b . However, since goods produced by type 1^i have the same liquidation value for type 1^j , as the good they produce, what matters is the net deposits $d^a - d^b$.

¹⁶ \bar{x}_2^a is an upper bound as by concavity of the utility function, $u(x_2^a) - x_2^a$ is decreasing beyond $x_2^a = x^*$.

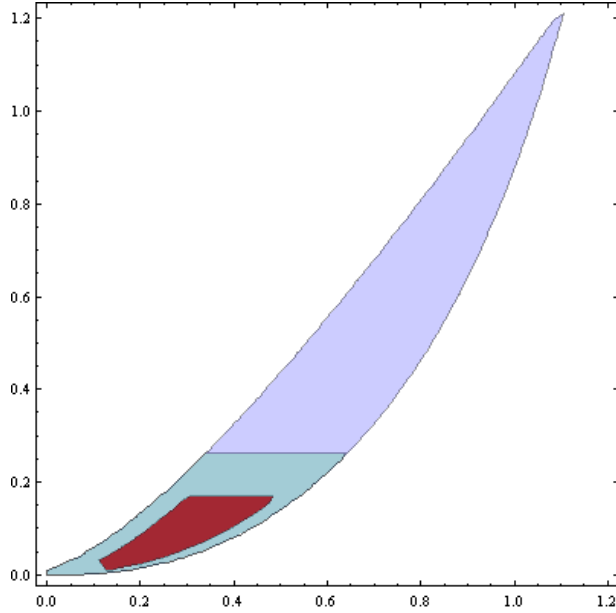


Figure 2: An example where deposits are essential.

Consider now again the example where all the ex ante gains from trade come from good 2 and where good 1 cannot be used to transfer utility across types, i.e. $U^1(x_1, x_2) = -x_2$ and $U^2(x_2, x_1) = u(x_2)$. To illustrate how deposits help despite assuming identical preferences across sectors, we assume $\lambda^a = \lambda^b = 1$. Therefore the only heterogeneity across sector here might come from the fact that agents in one sector are monitored or trade more often than in the other sector. However, for what follows, the two sectors can also be perfectly identical. For the moment, we will assume that the planner seeks to expand the set of IF allocations in sector a although we do not spell out the reasons why this is the case. From the analysis in the previous section, we know that the dynamic participation constraint for type 2^b (12) is never binding, and similarly for type 2^a . Therefore the only relevant constraints defining IF allocations in sector b are

$$-x_2^b - t + \theta^b [u(x_2^b) - x_2^b - t] \geq 0 \quad (22)$$

$$-x_2^b - d + \theta^b [u(x_2^b) - x_2^b - t] \geq 0 \quad (23)$$

And as the planner sets $t = \hat{t}$, we obtain

$$-\hat{x}_2^b - \hat{t} + \theta^b [u(\hat{x}_2^b) - \hat{x}_2^b - \hat{t}] = 0 \quad (24)$$

$$-\hat{x}_2^b - d + \theta^b [u(\hat{x}_2^b) - \hat{x}_2^b - \hat{t}] \geq 0 \quad (25)$$

where \hat{t} and \hat{x}_2^b are defined by (15). Notice that we get an upper bound on the level of deposits by using the first equation to replace $\theta^b [u(\hat{x}_2^b) - \hat{x}_2^b - \hat{t}]$ in the second

$$d \leq \hat{t} = \frac{\theta^b}{1 + \theta^b} u(\hat{x}_2^b) - \hat{x}_2^b.$$

Hence $d = \hat{t}$ is the biggest deposit that type 1^b can accept when they already transfer \hat{t} to sector a . The intuition is that redeeming one unit of deposit is as costly as producing one unit of transfer (since the foregone utility of redeeming is $\lambda = 1$). At $d = \hat{t}$, the constraints in sector b are tight, however the ones in sector a are most relaxed. Since the dynamic participation constraint for type 2^a is never binding, IF allocations in sector a are defined by the dynamic participation and the repayment constraints of type 1^a

$$-x_2^a + \hat{t} + \theta^a [u(x_2^a) - x_2^a + \hat{t}] \geq 0 \quad (26)$$

$$-x_2^a + d + \theta^a [u(x_2^a) - x_2^a + \hat{t}] \geq 0 \quad (27)$$

Since $d \leq \hat{t}$ only the repayment constraint (27) is binding. Therefore it is obvious that increasing d from zero to $d > 0$ expands the set of IF allocations in sector a .

Here we have adopted the view that the planner's objective was to expand the set of feasible allocations in sector a . However, there is no good reason why the planner would not want to expand the set of feasible allocations in sector b instead. The planner would then adopt the same strategy of maximizing transfers from sector a to sector b and then set deposits equal to the transfers. Interestingly, if $\theta^b \geq \theta^a$ we obtain $\hat{t}^b \geq \hat{t}^a$.¹⁷ Using $U^1(x_1, x_2) = -x_2$ and $U^2(x_2, x_1) = \sqrt{x_2}$, we plot the set of

¹⁷Indeed we get

$$\begin{aligned} \frac{\partial \hat{t}}{\partial \theta} &= \frac{1}{(1 + \theta)^2} u(\hat{x}_2) + \frac{\theta}{1 + \theta} u'(\hat{x}_2) - 1 \\ &= \frac{1}{(1 + \theta)^2} u(\hat{x}_2) > 0. \end{aligned}$$

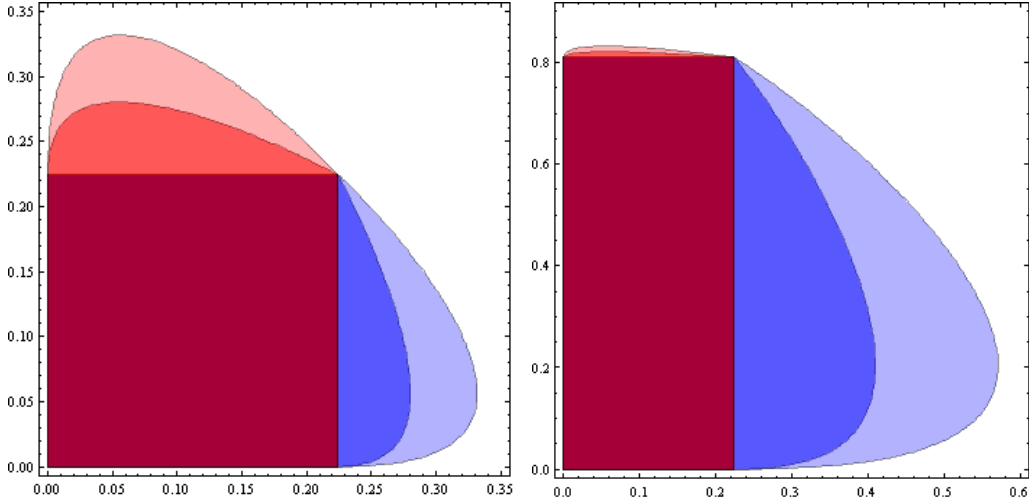


Figure 3: Left: $\theta^i = 0.9$ for $i = a, b$. Right: $\theta^a = 0.9$ and $\theta^b = 9$.

feasible allocations in the (x_2^a, x_2^b) -space for the cases when 1) $t^i = d^i = 0$ for all i in dark red 2) $t^i > 0$ and $d^i = 0$ for $i = a$ in medium red or $i = b$ in medium blue, and 3) $t^i, d^i > 0$ for $i = a$ or b , in light red or light blue respectively, in the case of homogenous or heterogeneous sectors.

The figure shows that nicely that the set of IF allocations is expanded relatively more in the sector with the relatively lower stake θ , and this whether only transfers are used or transfers and deposits. When only transfers are used (dark red plus medium red or medium blue zones), there are essentially two reasons that explain this aspect. First, as $\theta^b > \theta^a$, relatively more can be extracted from sector b to transfer to sector a . Second, because the utility function is concave, a same amount transfer have different effect across sectors: since agents in sector b can already sustain a relatively big amount of production on their own, they don not value much the extra consumption that some transfers would allow. And reversely, because agents in sector a cannot sustain much on their own, they will value relatively more the additional consumption that transfer enables. The intuition is slightly different when it comes to deposits. Since $\theta^b > \theta^a$ agents in sector b have more to lose if they liquidate storage.

where the last equality follows from the definition of \hat{x}_2 as the value for which $u'(\hat{x}_2) = (1 + \theta) / \theta$.

So, more can be stored with them than with agents in sector a . As a consequence, more can be produced in sector a and then stored in sector b than the reverse. This explains why the light red area is smaller than the light blue area when $\theta^b > \theta^a$.

In the previous example deposits alone cannot expand the set of IF allocations. Indeed, suppose for a moment $t = 0$. A quick examination of (26) and (27) is enough to realize that the dynamic participation constraint (26) always implies the repayment constraint (27) for any $d \geq 0$. This is intuitive: absent transfers, type 1^a are not immediately compensated for their production in sub-period 1, but only through future payoffs. Deposits do not help weakening this constraint, as they don't give instant return. This is however only an artefact of our assumption that type 1 do not consume any good 1. As we show in the example below, deposits can be essential (in the sense of Theorem 1) even when transfers are absent. Actually, we show deposits alone can do better than transfers alone.

As a last example, suppose type 2 are now endowed with a unit of good 1 instead of with a production technology for good 1. Let $U^1(1, x_2) = \bar{u} - x_2$ and $U^2(x_2, x_1) = u(x_2)$.¹⁸ We also assume as before $\gamma^a = \gamma^b$, $\lambda^a = \lambda^b = 1$ and $\theta^b \geq \theta^a$. In this case, the relevant constraints defining the set of IF allocations in sector b are

$$\begin{aligned} \bar{u} - x_2^b - t + \theta^b [\bar{u} + u(x_2^b) - x_2^b - t] &\geq 0 \\ -x_2^b - d + \theta^b [\bar{u} + u(x_2^b) - x_2^b - t] &\geq 0 \end{aligned}$$

When the planner only uses transfers so that $d = 0$, it can extract from sector b at most \hat{t} , where

$$\hat{t} = \frac{\theta^b}{1 + \theta^b} u(\hat{x}_2^b) + \bar{u} - \hat{x}_2^b.$$

and as before, \hat{x}_2^b is defined as the amount of good 2 that maximizes the transfer to sector a . Absent transfers, the planner can store at most \hat{d} with sector b where

$$\hat{d} = \theta^b [u(\hat{x}_2^b) + \bar{u}] - (1 + \theta^b) \hat{x}_2^b.$$

where \hat{x}_2^b is the amount of good 2 that maximizes deposits from sector a (and it happens to be the amount of good 2 that would maximize transfer to sector a , which

¹⁸We assume the planner always transfers all the endowment of good 1 to type 1.

will simplify the analysis later on). We can carry out the analysis when $\bar{u} < \hat{d}$ but it requires some additional constraints and therefore, for ease of exposition, we only consider here the case where $\bar{u} \geq \hat{d}$. Let us now consider how the constraints in sector a are relaxed when transfers alone or deposits alone are used. We know that the production constraint for agents 2 never binds in any case. In the case where transfers alone are used, constraints (6) and (16) in sector a become

$$\bar{u} - x_2^a + \hat{t} + \theta^a [\bar{u} + u(x_2^a) - x_2^a + \hat{t}] \geq 0 \quad (28)$$

$$-x_2^a + \theta^a [\bar{u} + u(x_2^a) - x_2^a + \hat{t}] \geq 0 \quad (29)$$

Clearly, only (29) binds, and replacing the value for \hat{t} , it is

$$-x_2^a + \theta^a [\bar{u} + u(x_2^a) - x_2^a] + \theta^a \left[\frac{\theta^b}{1 + \theta^b} u(\hat{x}_2^b) + \bar{u} - \hat{x}_2^b \right] \geq 0 \quad (30)$$

In the case where only deposits are used, the dynamic participation and repayment constraints become

$$\bar{u} - x_2^a + \theta^a [\bar{u} + u(x_2^a) - x_2^a] \geq 0 \quad (31)$$

$$-x_2^a + d + \theta^a [\bar{u} + u(x_2^a) - x_2^a] \geq 0 \quad (32)$$

Since $d \leq \hat{d} \leq \bar{u}$, (32) implies (31). Also, as any feasible deposits have to satisfy $d \leq x_2^a$ the set of IF allocations is expanded most if

$$d = \min\{x_2^a, \hat{d}\}$$

For allocations where $x_2^a < \hat{d}$, (32) is equivalent to IR and is therefore redundant. In cases where $\hat{d} < x_2^a$, the repayment constraint becomes

$$-x_2^a + \theta^a [\bar{u} + u(x_2^a) - x_2^a] + \theta^b \left[u(\hat{x}_2^b) + \bar{u} - \frac{1 + \theta^b}{\theta^b} \hat{x}_2^b \right] \geq 0 \quad (33)$$

Clearly, if θ^a is low enough, (30) implies (33) and using only deposits implement more allocations than using only transfers. The figure below shows an example for some arbitrary \bar{u} . It shows the set of IF allocations in the (x_2^a, x_2^b) -space for the cases when 1) $t = d = 0$ in dark red 2) $t > 0$ and $d = 0$ in medium red, and 3) $d > 0$ and $t = 0$ in light red.

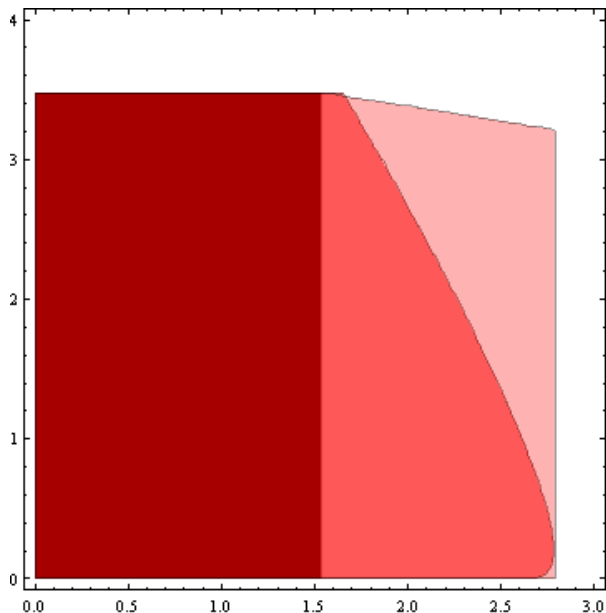


Figure 4: Fixed endowment.

Absent transfers, the set of IF allocations x_2^a does not depend on x_2^b . In other words, the upper bound \bar{x}_2^a on the set of IF allocations is independent on x_2^b . This is not the case in sector b as deposits from sector a influence directly the set of incentive feasible allocations. The figure is somewhat deceptive as it is not necessarily the case that the upper bound on the feasible allocations in sector a with transfer alone equals the one using deposits. Finally, let us mention that for low value of \bar{u} , some allocations are feasible only with transfers and are not feasible with deposits alone. Indeed, for $\bar{u} = 0$, our previous example shows that only transfers work.

5 Bankers

We have established that deposits are essential. However, we have not taken a stance on who should take deposits. To address this, we assume that agents have different probabilities of gaining from market activities in a given period – i.e. they have different stakes in the economic system. Even with equal monitoring probabilities, those with a higher stake in the system are less inclined to deviate from proscribed

behavior. Consistent with experience, individuals with a greater connection to the market are better suited to play the role of bankers, since they have more to lose by renegeing on obligations. Hence, in the sequel, we assume that agents in sector b have a higher probability of trading $\gamma^b > \gamma^a$, while the monitoring probability is the same across sectors $\pi^a = \pi^b$. In the next section, we discuss how to endogenize π^i .

The question of who should accept deposits, or who should be banker, is normative, and we need to introduce a selection criterion to answer it. There is no good reason why the planner would favor one sector over the other, and as a first step we will therefore resort to a criterion related to Pareto optimality. An allocation is Pareto optimal if there is no other allocation that a subset of agent prefers while the complementary subset is at least indifferent. In a given sector, the set of Pareto optimal allocations is defined as those allocations x that are the solution to the following problem,

$$\max_x \mathcal{W}(x; \gamma) = \max_x \gamma [\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1)] + \frac{\theta}{\pi} S(x_1, x_2) \quad (34)$$

with the Pareto weights ω_1, ω_2 such that $\omega_1 + \omega_2 \leq 1/\gamma$. In other words, $\mathcal{W}(x^i; \gamma^i)$ is our selection criterion within sector i . Let \mathcal{F}_0^i be the set of IF allocations in sector i with *no* transfers and deposit $d = 0$. Because of the usual conditions, \mathcal{F}_0^i is a convex set. We are interested in the best IF allocation in sector i , according to our selection criterion, i.e. in the allocation \hat{x}^i that solves

$$\max_{x^i} \mathcal{W}(x^i; \gamma^i), \text{ s.t. } x^i \in \mathcal{F}_0^i.$$

Clearly, by definition, there is no IF allocation in sector i different from \hat{x}^i that can make one agent better off without hurting the other agent. Now, given \hat{x}^a and \hat{x}^b and our selection criterion \mathcal{W} , we want to know whether using transfers and/or deposits in sector i , the planner can choose another allocation that would make sector j better off than at \hat{x}^j , without making sector i worse off than at \hat{x}^i . Clearly, transfers do not help, as the sector making the transfer is made worse off. If deposit can help, then we will say that deposits in sector j are Pareto essential. Deposits are essential as

they increase the set of IF allocations in sector i and they are Pareto essential as they increase \mathcal{W} in sector i . More precisely, we will say that *deposits in sector i are Pareto essential* if there is an allocation $(\tilde{x}^a, \tilde{x}^b)$ and deposits $d > 0$ in sector i such that $\mathcal{W}(\tilde{x}^j; \gamma^j) \geq \mathcal{W}(\hat{x}^j; \gamma^j)$ for $j = a, b$ with at least one strict inequality in one sector. Notice already that a necessary condition for Pareto essentiality of deposits in sector i is that the repayment constraint does not bind at \hat{x}^i . Otherwise, strictly positive deposits in sector i will make the repayment constraints tighter, thus shrinking the set of IF allocations in sector i .

We think Pareto essentiality is an appropriate criterion for one simple reason (that we, however, do not model). If agents in the sector with depositors were better off without deposits, then we could well imagine that this sector would revert to the best allocation with no deposits. Similarly, if those agents in the sector with bankers, i.e. who take deposits, are hurt by doing so, there is no reason to think that banks would emerge endogenously. In the Appendix, we show the following result:

Theorem 2 *Suppose $\gamma^b > \gamma^a$. Then only deposits in sector b are Pareto essential.*

Notice that we do not make any specific assumption on agents' preferences other than the one already mentioned. In particular, we do not assume that preferences are represented by a separable or quasi-linear utility function. The spirit of the proof is best illustrated by Figure 6 below. The set in light blue is the set of IF allocation in sector b , while the orange set is the set of IF allocations in sector a . Since $\gamma^b > \gamma^a$, the latter set is strictly contained in the former. The point x^* is the solution to the unconstrained planner problem, given weights λ_1 and λ_2 . The graph illustrates an example where $\lambda_2 > \lambda_1$. The Pareto frontier is always monotone decreasing, although it can be concave or convex. Maybe more surprising, we show the optimal allocation x^* is the same across sectors, as long as agents in both sectors have the same preferences. The figure also shows some indifference curves from the planner's problem, one going through \hat{x}^a and the other going through \hat{x}^b . An important aspect of the problem is that the objective function of the planner increases as the indifference curve gets closer to x^* .

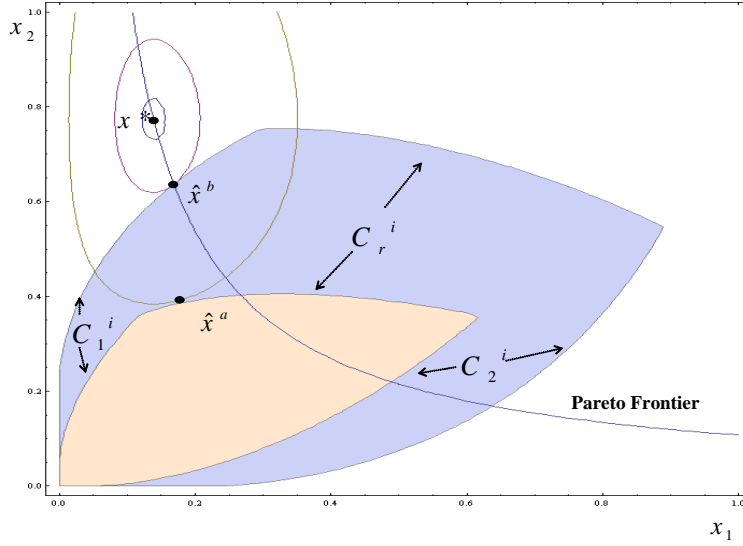


Figure 5: Deposits in sector b are Pareto essential.

While the figure represents the case where x^* is not feasible in either sector, this needs not be the case in general. If x^* is feasible in sector a then it is also feasible in sector b since $\gamma^b > \gamma^a$, but not vice-versa. In the case where x^* is feasible in sector b , agents in this sector are already at bliss and deposits in sector a cannot have any role. To the contrary, deposits in sector b have a role if, for instance, the repayment constraint binds in sector a at \hat{x}^a . In this case, deposits can relax the constraint, so the planner can select an allocation that is closer to x^* .

Now let us return to the case where x^* is not feasible in either sector, as shown in the figure. The incentive feasible set is characterized by three curves in each sector $i = a, b$. The dynamic participation constraint for type 1, C_1^i , the one for type 2, C_2^i and finally the repayment constraint for type 1, C_r^i . We can show that if the repayment constraint is binding in sector b , that is if \hat{x}^b is on the curve C_r^b , then it must be the case that the repayment constraint is also binding in sector a , that is \hat{x}^a is also on the curve C_r^a . But, as shown in the figure, the reverse is not true. Therefore, it can never be the case that deposits in sector a are used to relax the repayment constraint in sector b without making agents in sector a worse off. We want to mention here that the result is not as trivial as it might first appear, since

we need to rule out the case where \hat{x}^a is on the curve C_1^a while \hat{x}^b is on C_r^b . We do this showing that if \hat{x}^a is on the curve C_i^a for $i = 1$ or 2 , then it also is on the Pareto frontier.

It is also worth mentioning one general property of the solution \hat{x} . Whenever one dynamic participation constraint binds, the Pareto frontier must intersect this participation constraint, and the solution lies at the intersection of this constraint with the Pareto frontier. In the case where the repayment constraint bind, the solution is not on the Pareto frontier. To see this it is easiest to consider the case where $\pi = 1$. In this case, each dynamic participation constraint is related to the indifference curve for the relevant agent. For instance if the dynamic participation constraint for agent i is binding, then agent i gets a zero payoff. However, by definition, the Pareto frontier describes the point of tangency for the two indifference curves. Therefore, ignoring the repayment constraint, the solution will be on the Pareto frontier and on a dynamic participation constraint. However, for θ^i low enough, the repayment constraint will always bind in sector i and deposits in sector j might have a role. This is important for the result in the next section.

An alternative to using the concept of Pareto essentiality is to maximize the expected utility of an agent under the veil of ignorance, i.e. when the agent does not know whether he will live in sector a or in sector b . To illustrate how this criterion works, we use the example where x_1 is not produced or consumed in either sector so that $U^1(x_1, x_2) = -x_2$ and $U^2(x_2, x_1) = u(x_2)$. Also, we assume $\lambda^a = \lambda^b = 1$ and $\gamma^b > \gamma^a$ so that $\theta^b \geq \theta^a$. We also assume that each agent faces the same probability to live in either sector. Since agents are identical ex-ante, the lifetime expected payoff of a representative agent is proportional to

$$\gamma^a [u(x_2^a) - x_2^a] + \gamma^b [u(x_2^b) - x_2^b]$$

If $(x_2^a, x_2^b) = (x^*, x^*)$ is feasible without transfers or deposits, then it is the solution and neither transfers nor deposits are used. From now on, we suppose the efficient allocation x^* is not feasible in sector a . As $\theta^b \geq \theta^a$ we know that x^* can be feasible in sector b although it is not feasible in sector a . Hence, we consider two cases, first, x^*

is feasible in sector b but not in sector a ,¹⁹ and x^* is not feasible in either sector. In the previous section, we showed that, given an allocation x_2^i , the maximum feasible transfers from and feasible deposit size in sector i given allocation x_2^i are

$$d^i = t^i = \frac{\theta^i}{1 + \theta^i} u(x_2^i) - x_2^i.$$

Notice that this is also the resource that is necessary to implement x_2^i in case the dynamic participation is not satisfied at x_2^i . Feasibility requires $\gamma^a d^a + \gamma^b d^b \geq 0$ and $\gamma^a t^a + \gamma^b t^b \geq 0$. Therefore, the planner's problem is simply

$$\begin{aligned} \max_{x_2^a, x_2^b} & \gamma^a [u(x_2^a) - x_2^a] + \gamma^b [u(x_2^b) - x_2^b] \\ \text{s.t.} & \gamma^a \left[\frac{\theta^a}{1 + \theta^a} u(x_2^a) - x_2^a \right] + \gamma^b \left[\frac{\theta^b}{1 + \theta^b} u(x_2^b) - x_2^b \right] \geq 0. \end{aligned} \quad (35)$$

If the constrained efficient allocation features a negative surplus for producers in sector i , then notice that $d^i = t^i < 0$. In words, the constrained efficient allocation will recommend deposits in sector j and transfers to sector i . Notice that a transfer from sector i to sector j is not affecting welfare from an ex ante point of view since $\lambda = 1$. The first order conditions of (35) give

$$\frac{1 + \theta^a}{u'(x_2^b) - 1} - \frac{1 + \theta^b}{u'(x_2^a) - 1} = \theta^b - \theta^a$$

When x^* is not feasible and both sectors are homogenous, i.e. $\theta^a = \theta^b = \theta$, we know that the constraint binds. The constrained efficient allocation is then $x_2^a = x_2^b = \bar{x}_2$, where \bar{x}_2 is the maximum level of production in sector a without transfers or deposit. Producers have no surplus at \bar{x}_2 as it satisfies

$$u(\bar{x}_2) = \frac{1 + \theta}{\theta} \bar{x}_2.$$

If $\theta^b > \theta^a$, then the constrained efficient allocation fully insures agents ex ante, but not ex post as $x_2^b > x_2^a$. Loosely speaking, there is insurance as the allocations with transfers and deposit across sectors are 'closer' to one another. It is rather intuitive

¹⁹i.e. $-x^* + \theta^b [u(x^*) - x^*] \geq 0$, but $-x^* + \theta^a [u(x^*) - x^*] < 0$.

how using transfers and deposits can bring the allocations in both sectors closer. For example, in the case where x^* is feasible in sector b , the planner can use transfers and deposits to increase x_2^a while leaving x_2^b unchanged. Actually, we show in the Appendix that, at the Pareto optimal allocation, the planner will not use deposits or transfers in sector a as this would instead increase the allocations' wedge across the two sectors.

5.1 Endogenous Monitoring

In the previous section, we studied the case where the monitoring probability was exogenous. We now ask which agents should be endogenously monitored and how much they should be monitored, when monitoring is costly. To simplify the analysis, we assume that the probability to monitor production is fixed to some level $\pi \in [0, 1]$ in both sector. However the planner can now choose the probability to monitor repayment in each sector. We will let $\tilde{\pi}^i$ denote the monitoring probability in sector i . Given γ^i is fixed, choosing $\tilde{\pi}^i \in [0, 1]$ is equivalent to choosing $0 \leq \tilde{\theta}^i \leq \beta\gamma^i / (1 - \beta)$. We assume that monitoring an agent in either sector at a level $\tilde{\pi}$ requires an investment of $\beta\tilde{\pi}K > 0$ each period.²⁰ Since only type 1 are monitored and there is a measure γ^i of these agents in each period, the lifetime costs from monitoring repayment is $\tilde{\pi}^i\gamma^i\beta K / (1 - \beta) = \tilde{\theta}^i K$. An allocation in sector i with no deposit is now a pair $(x^i, \tilde{\theta}^i)$ that solves

$$\max_{(x^i, \tilde{\theta}^i)} \mathcal{W}(x^i; \gamma^i) - \tilde{\theta}^i K, \text{ s.t. } x^i \in \tilde{\mathcal{F}}_0^i \text{ and } \tilde{\theta}^i \leq \beta\gamma^i / (1 - \beta), \quad (36)$$

where $\mathcal{W}(x^i; \gamma^i)$ satisfies (34) and $\tilde{\mathcal{F}}_0^i$ is defined as before except that the repayment constraint is instead

$$-\lambda x_2^i + \tilde{\theta}^i S(x_1^i, x_2^i) \geq 0. \quad (37)$$

At the solution to the planner's problem (37) is binding, as otherwise the planner would be able to reduce the monitoring probability slightly. It should be clear to

²⁰Assume for simplicity that this monitoring cost is paid up front, before the economy starts.

the reader that with endogenous monitoring, x^* is never a solution to the planner's problem. The reason is intuitive: Suppose the planner's solution is $x^*, \tilde{\theta}$. Then reducing the monitoring probability slightly is a first order gain, while moving away from x^* is a second order loss.

We now ask, given the best allocation in sector i with no deposit is $(\tilde{x}^i, \tilde{\theta}^i)$, are there levels of deposit and monitoring that are Pareto essential? The answer is now more delicate than before: since monitoring is a costly choice, the repayment constraint will always bind at the solution $(\tilde{x}^i, \tilde{\theta}^i)$. Otherwise, the planner would be able to reduce the monitoring probability slightly and save on monitoring costs. Given the repayment constraint always bind, it is never possible to make one sector better off without making the other worse off. Still, there may be gains from monitoring one sector more than the other. In particular, it may be desirable to deposit goods in sector i say. However, this implies that monitoring has to be increased in sector i . This is desirable if it is cheaper for agents in sector j to compensate the cost of the extra monitoring in sector i , than to do without deposits and monitor their type 1 agents instead. Precisely, we will say that a level of deposit d in sector i is monitoring-feasible if the new monitoring level $\tilde{\pi}^{i'} > \tilde{\pi}^i$ that accomodate d in sector i allows a reduction of monitoring in sector j from $\tilde{\pi}^j$ to $\tilde{\pi}^{j'}$ such that the overall monitoring cost does not increase, $\tilde{\theta}^{j'} + \tilde{\theta}^{i'} < \tilde{\theta}^j + \tilde{\theta}^i$.

Theorem 3 *Suppose $\gamma^b > \gamma^a$. Then only deposits in sector b are monitoring-feasible.*

Proof. Since $\gamma^b > \gamma^a$, it must be that $S(\tilde{x}_1^b, \tilde{x}_2^b) \geq S(\tilde{x}_1^a, \tilde{x}_2^a)$, as what is feasible in sector a is also feasible in sector b . Introducing deposit d from sector a to sector b , the repayment constraint in sector b becomes (it holds with equality at any solution)

$$-\lambda \left(x_2^b + \frac{\gamma^a}{\gamma^b} d \right) + \tilde{\theta}^b S(x_1^b, x_2^b) = 0.$$

And any increase in d increases the monitoring cost by

$$\frac{\partial \tilde{\theta}^b}{\partial d} = \frac{\gamma^a}{\gamma^b} \frac{\lambda}{S(\tilde{x}_1^b, \tilde{x}_2^b)}.$$

Now, the repayment constraint in sector a is

$$-\lambda(x_2^a - d) + \tilde{\theta}^a S(x_1^a, x_2^a) = 0.$$

so that increasing d reduces the monitoring cost in sector a by

$$\frac{\partial \tilde{\theta}^a}{\partial d} = -\frac{\lambda}{S(\tilde{x}_1^a, \tilde{x}_2^a)}.$$

Therefore, increasing deposits from sector a to sector b reduces the overall monitoring cost $\tilde{\theta}^a + \tilde{\theta}^b$ since

$$\frac{\partial \tilde{\theta}^a}{\partial d} + \frac{\partial \tilde{\theta}^b}{\partial d} = \frac{\gamma^a}{\gamma^b} \frac{\lambda}{S(\tilde{x}_1^b, \tilde{x}_2^b)} - \frac{\lambda}{S(\tilde{x}_1^a, \tilde{x}_2^a)} < 0$$

where the inequality follows from the fact that $S(\tilde{x}_1^a, \tilde{x}_2^a) \leq S(\tilde{x}_1^b, \tilde{x}_2^b)$ and $\gamma^a < \gamma^b$. Hence, given $d = 0$ when sectors are not using deposits, only $d > 0$ is monitoring feasible (and not $d < 0$). ■

The proof illustrates why only deposits in sector b are monitoring-feasible. A first effect plays through the number of traders in sector b : since $\gamma^b > \gamma^a$, deposits from sector a are spread among more agents. Automatically, this lowers the monitoring probability in sector a more than it is increased in sector b . A second, more interesting, effect is that agents in sector b have more at stake, since $\gamma^b > \gamma^a$ and $S(\tilde{x}_1^b, \tilde{x}_2^b) \geq S(\tilde{x}_1^a, \tilde{x}_2^a)$. Therefore, a relatively lower increase in monitoring probability is enough to induce the right behaviour in sector b . As a corollary we get that if γ^b is large enough then there will be no monitoring of repayment in sector a as all the production will be deposited with sector b .

Corollary 4 *Suppose $\gamma^b > \bar{\gamma}$ defined in the proof. Then only sector b is monitored for repayment, i.e. $\tilde{\theta}^a = 0$.*

Proof. Let \check{x}^a solve

$$\begin{aligned} \max_{x^a} \mathcal{W}(x^a; \gamma^a), \text{ s.t. } U^1(x_1^a, x_2^a) + \theta^a S(x_1^a, x_2^a) &\geq 0, \text{ and} \\ U^2(x_2^a, x_1^a) + \theta^a S(x_1^a, x_2^a) &\geq 0 \end{aligned}$$

If there is $\tilde{\theta}^b$ such that

$$\tilde{\theta}^b = \frac{\lambda(\tilde{x}_2^b + \gamma^a \tilde{x}_2^a / \gamma^b)}{S(\tilde{x}_1^b, \tilde{x}_2^b)} < \gamma^b \frac{\beta}{1 - \beta}$$

then it is optimal to set $d = \tilde{x}_2^a$ and $\tilde{\theta}^a = 0$. $\bar{\gamma}$ is then defined by the last inequality.

■

5.2 Optimal Measure of Bankers

In this section we ask how many bankers are desirable. The answer hinges on the following tradeoff: having fewer bankers entails lower monitoring costs, but leads to more deposits per banker, which increases incentives to misbehave. It may not be optimal to have all agents of type 1 in one sector to be bankers. Therefore we need to take a stance regarding the monitoring probability of those agents who are not bankers in a given sector. We assume here that those agents are monitored according to the intensity that solves (36).²¹

Since we still assume $\gamma^b > \gamma^a$, bankers are in sector b and we let B denote the measure of agents 1^b who are bankers. Bankers are monitored with probability $\dot{\pi}^b$ and we will denote their connection to the market by $\dot{\theta}^b$. Agents of type 1 in sector a are monitored with probability $\dot{\pi}^a$ with a market connection of $\dot{\theta}^a$. The monitoring probabilities $\dot{\pi}^b$ and $\dot{\pi}^a$ are defined using the repayment constraints for each agents, and we can get the following expressions for $\dot{\theta}^a$ and $\dot{\theta}^b$,

$$\dot{\theta}^a = \frac{\lambda(x_2^a - d)}{S(x_1^a, x_2^a)} \quad (38)$$

and

$$\dot{\theta}^b = \frac{\lambda(\gamma^b B x_2^b + \gamma^a d)}{\gamma^b B S(x_1^b, x_2^b)}. \quad (39)$$

Notice that if $d = 0$, then the expression for $\dot{\theta}^b$ is identical to the one when sectors are independent. Also, notice that as $\dot{\pi}^b \leq 1$, it is necessarily the case that $\dot{\theta}^b \leq$

²¹Alternatively, we could assume that these agents can also deposits some goods with bankers in their sector.

$\gamma^b\beta/(1-\beta)$. Given an allocation (x^a, x^b) , the planner seeks to minimize the cost of monitoring, or

$$\min_{B \in [0,1], \dot{\theta}^b \leq \gamma^b\beta/(1-\beta), \dot{\theta}^a, d} \gamma^b B \dot{\theta}^b + \gamma^b (1-B) \tilde{\theta}^b + \gamma^a \dot{\theta}^a \quad \text{s.t. (38) and (39)}$$

Replacing (38) and (39) in the objective function, we obtain

$$\gamma^b \tilde{\theta}^b + \gamma^a \frac{\lambda x_2^a}{S(x_1^a, x_2^a)} + \lambda \gamma^a d \left[\frac{1}{S(x_1^b, x_2^b)} - \frac{1}{S(x_1^a, x_2^a)} \right].$$

When $d = 0$, we know the solution to the planner is described by the solution to (36), so that total is described by $\gamma^b \tilde{\theta}^b + \gamma^a \tilde{\theta}^a$. Keeping the allocation \tilde{x}^a unchanged, the planner can increase d to save on monitoring costs in sector a , as described by the term in square bracket.²² The planner can also modify the allocation in sector a to be closer to bliss, at the cost of increasing monitoring cost in sector a . Given an allocation such that $S(x_1^b, x_2^b) > S(x_1^a, x_2^a)$, it should be clear that the planner prefers to set d as high as possible.²³ In particular, from (39), we have

$$\lambda \gamma^a d = B \gamma^b \dot{\theta}^b S(x_1^b, x_2^b) - B \lambda \gamma^b x_2^b. \quad (40)$$

The planner therefore needs to set $\dot{\theta}^b > \tilde{\theta}^b$ and $B > 0$. However, in this linear world, both parameters work together. For instance, the planner can increase $\dot{\pi}^b$ to 1 and let B determined by (40) for a given level of d , or inversely. Finally, if

$$\lambda \gamma^a x_2^a < \gamma^b \frac{\gamma^b \beta}{1-\beta} S(x_1^b, x_2^b) - \lambda \gamma^b x_2^b$$

$d = x_2^b$ is feasible. In this linear cost environment, the planner has multiple choice and can set $B < \dot{\pi}^b = 1$ or $\dot{\pi}^b < B = 1$ or $\dot{\pi}^b < 1$ and $B < 1$. Still there is a tradeoff between the number of bankers and the monitoring intensity.

5.3 Positive Rate of Return on Storage

In this section, we show using a simple example, that bankers can be in a sector with a relatively inefficient storage technology, as long as the commitment problem in the

²² As $S(\tilde{x}_1^b, \tilde{x}_2^b) > S(\tilde{x}_1^a, \tilde{x}_2^a)$.

²³ In the previous section, we show that $S(x_1^b, x_2^b) > S(x_1^a, x_2^a)$ is a necessary condition for deposits in sector b to be monitoring feasible, given allocation (x^a, x^b) .

other sector is severe enough. In other words, bankers are not necessarily the one with the better storage technology, as bankers also need a relatively high degree of commitment. To do so, we consider once again the case where good 1 is absent and all gains from trade come from good 2, i.e. $U^1(x_1, x_2) = -x_2$ and $U^2(x_2, x_1) = u(x_2)$. Also, we assume $\gamma^a = \gamma^b$ and $\lambda^a = \lambda^b = 1$ and that the planner puts equal weights on agents of type 1 and type 2, i.e. $\omega_1 = \omega_2 = \omega$. Finally, we denote by ρ^i the net return on storage in sector i , so that for each unit stored in sector i , the technology returns $1 + \rho^i$. We assume that the storage technology is better in sector a , that is $\rho^a > \rho^b$. In addition we set $\rho^b = 0$ and to ease notation we set $\rho^a = \rho$. Our goal is to show that for some parameters, deposits in sector b are Pareto essential, in spite of a more efficient storage technology in sector a .

When there is no good 1, we know from our earlier analysis that the incentive constraints for agents of type 2 never bind. Hence, absent any interaction between the two sectors and given $\omega_1 = \omega_2$ the planner maximizes the trade surplus in each sector, subject to the production and repayment constraints,

$$\max_{x_2^i} u(x_2^i) - \frac{x_2^i}{1 + \rho^i} \quad (41)$$

s.t.

$$-\frac{x_2^i}{1 + \rho^i} + \theta^i \left[u(x_2^i) - \frac{x_2^i}{1 + \rho^i} \right] \geq 0 \quad (42)$$

$$-x_2^i + \theta^i \left[u(x_2^i) - \frac{x_2^i}{1 + \rho^i} \right] \geq 0 \quad (43)$$

Ignoring incentive constraints, the first best solution is given by $x_2^{*,i}$ solving $u'(x_2^{*,i})(1 + \rho^i) = 1$. As $\rho > 0$ we obtain $x_2^{*,a} > x_2^{*,b}$. Notice that storage gives additional units at no extra cost, and as its rate of return increases, agents of type 1 can reduce their production to sustain a given level of consumption x_2^i . Therefore when $\rho^i > 0$, only the repayment constraint (43) is relevant and the production constraint can be ignored. We denote by \underline{x}_2^a the level of x_2^a that satisfies the repayment constraint (43) as an equality. Then \underline{x}_2^i is the largest feasible consumption in sector i , when there is no connection between the two sectors. We can then define the level of market connection $\bar{\theta}^i$ for which $\underline{x}_2^i = x_2^{*,i}$. That is, for any $\theta^i \geq \bar{\theta}^i$ the repayment constraint is satisfied even

at $x_2^{*,i}$, and $x_2^{*,i}$ is then the solution to the planner's problem. Similarly, if $\theta^i < \bar{\theta}^i$, the repayment constraint is violated at $x_2^{*,i}$, and therefore sector i cannot be at bliss without some help from the other sector.

As $\rho > 0$, observe that $\bar{\theta}^b > \bar{\theta}^a$ is a possible configuration. In this case, even if we would assume $\theta^b > \theta^a$, the repayment constraint could bind in sector b but not in sector a , if for example $\bar{\theta}^b > \theta^b > \theta^a > \bar{\theta}^a$. This is in sharp contrast with the previous analysis and it opens the door for many possible cases, each of which we analyze in the Appendix. Here we spend more time on the result of interest, namely cases where deposits in sector b are Pareto essential. In the Appendix, we show

Theorem 5 *Deposits in sector b are Pareto essential if $\theta^a < \bar{\theta}^a$ and*

$$\theta^b \geq \bar{\theta}^b \text{ and } \theta^a \rho < 1 + \rho,$$

or

$$\theta^b < \bar{\theta}^b \text{ and } \theta^a \rho < \left(1 - \frac{\theta^a}{\theta^b}\right) (1 + \rho).$$

The condition that $\theta^a < \bar{\theta}^a$ requires that agents in sector a are not at bliss. Obviously, if they were, deposits could not be an improvement. Then we have to consider the situation of agents in sector b . They are at bliss if $\theta^b \geq \bar{\theta}^b$. However, although agents in sector b do not have a commitment problem when $\theta^b \geq \bar{\theta}^b$, they still have a bad storage technology. Therefore increasing deposits in sector b requires agents in sector a to produce *more* to cover for the loss in storage return and sustain a given consumption level. Indeed, when the allocation in sector a is x_2^a and deposit is d , agents in sector a have to produce y , such that

$$x_2^a = (y - d)(1 + \rho) + d.$$

Agents in sector a earn ρ on their own storage $y - d$, however, they get zero net return when they store with sector b . Therefore they need to produce $y = x_2^a / (1 + \rho) + d\rho / (1 + \rho)$. To achieve the same consumption level x_2^a , one additional unit of deposits in sector b requires that $\rho / (1 + \rho)$ additional units be produced. Agents

discount this cost by the level of market connection θ^a , as they do not turn out to be of type 1 each period. The condition $\theta^a \rho / (1 + \rho) < 1$ insures that increasing deposits in sector b relaxes the repayment constraint in sector a : while using deposits increases production in sector a , it relaxes the repayment constraint enough that there is room for additional production and therefore getting closer to the bliss point.

The second case is similar, except that agents in sector b are now constrained since $\theta^b < \bar{\theta}^b$. Therefore, they need to be compensated for taking deposits, as otherwise they would default. A transfer from sector a does just that. However, it imposes an additional production cost on agents in sector a . We calculate that the minimum transfer to keep agents in sector b from defaulting is d/θ^b . Again, agents in sector a discount this additional production by θ^a , so that when $\theta^a \rho / (1 + \rho) + \theta^a / \theta^b < 1$, increasing deposits in sector b relaxes the repayment constraint in sector a , while keeping agents in sector b indifferent.

It is easy to see that, as the storage technology of agents in sector a becomes better, their commitment problem has to also become more severe for deposits in sector b to be Pareto essential. Still, the bottom line is that bankers are not necessarily the one with the better technology, as bankers also need a relatively high degree of commitment. Figure 6 illustrates our example when $u(x_2) = \sqrt{x_2}$ and $\rho = 0.5$, for the cases where $\theta^b > \theta^a$. The blue area shows parameters where deposits in sector b are Pareto-essential. The pink area shows the area where both sectors are already at bliss without using deposits. The white area above the 45°-line and below the blue area depicts the case where deposits in sector b are not Pareto essential because the return on storage is too large relative to the commitment issue in sector a .

Finally, in Figure 7, we show an example with $\rho = 2$, in which deposits in sector a are Pareto essential in spite of the fact that $\theta^a < \theta^b$. We refer the interested reader to the Appendix for details on the derivations. Notice that deposits do not become Pareto essential in sector b once $\theta^a < \bar{\theta}^a$. Rather, the commitment problem has to be sufficiently severe in sector a that it becomes beneficial to give up some of the return on storage and use deposits in sector b . Finally, as the commitment problem in sector

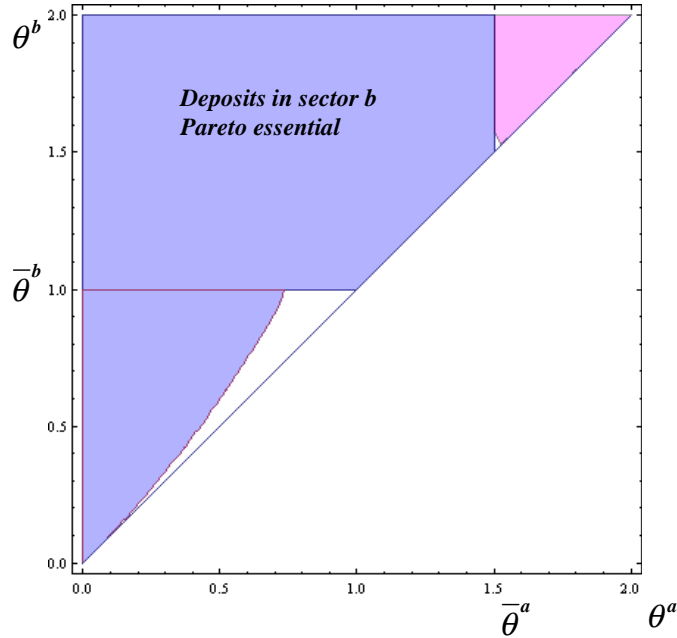


Figure 6: $\rho > 0$ and deposits in sector b are Pareto essential.

b becomes relatively more severe, deposits in sector a can become Pareto essential as the return on storage is so high. However, this is limited by the commitment problem in sector a . Notice that there are some parameters for which deposits are Pareto essential in both sectors.

5.4 Banknotes

So far, our analysis was based on the set of incentive feasible allocations and we did not dwell on how to implement these allocations. In this section, we illustrate the use of banknotes by considering the specific case where $\theta^a = 0$. Then, absent any mechanism, agents of type 1 in sector a would never deliver their storage to agents of type 2 in sector a . Bankers and deposits are part of such a mechanism. However the details will depend on the matching technology available to the planner.

First, suppose the planner is able to match a pair of type 1 and type 2 agents from sector a with another pair of type 1 and type 2 agents from sector b . Also assume that the matching technology is such that all agents in a match can observe the

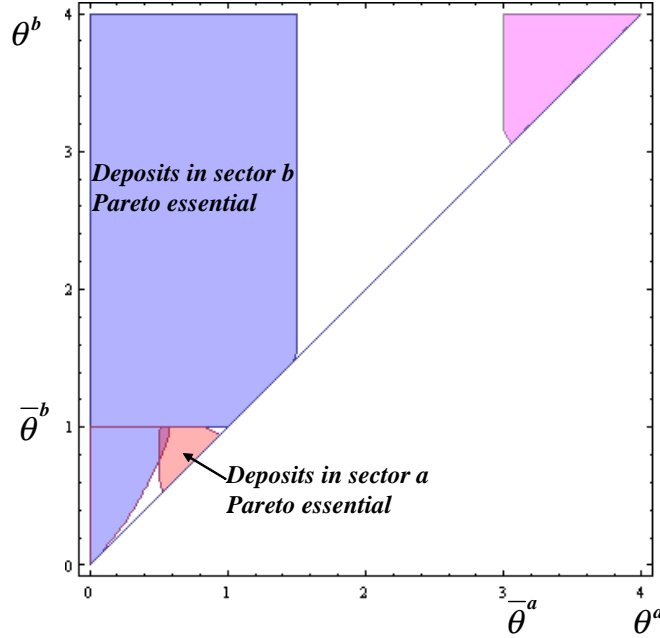


Figure 7: $\rho > 0$ and deposits in sector a and b are Pareto essential.

actions of all agents in the match in both period. Consider the following mechanism to implement a feasible allocation $(x_1^a, x_2^a, x_1^b, x_2^b)$. In the first subperiod, agent 1^a produces x_2^a and deposits it with agent 1^b . Only then does agent 2^a produce x_1^a and delivers it to agent 1^a and otherwise does not produce. In the second subperiod, agent 1^b delivers x_2^a to agent 2^a . This is incentive feasible as $\theta^b > 0$. This mechanism seems to work because agent 2^a can observe that x_2^a has been deposited with agent 1^b and therefore expects to consume it in the second subperiod. However, notice that it is beneficial for agent 2^a to deviate and not produce x_1^a . The reason is that in subperiod 2, agent 1^b will anyway deliver x_2^a to agent 2^a as otherwise he might suffer a punishment. The issue is that agent 2^a can exploit the fact that the production of x_2^a already occurred. Absent any other device, the only way to implement a feasible allocation (x_1^a, x_2^a) is that the production *and* storage of good x_2^a occurs simultaneously with the production and exchange of x_1^a .

Now, suppose the matching technology is such that agents can only meet pairwise (or alternatively, suppose that simultaneity as described above is not a feasible op-

tion). In other words, agents 1^a are unable to meet with agents 2^a and 1^b at the same time. Then simultaneity is not feasible and another device is necessary to implement an allocation. Naturally, we introduce banknotes, as objects that can be produced by type 1^b . The following mechanism can then implement a feasible allocation. First agent 1^a produce x_2^a and deposit it with agent 1^b in exchange for a banknote. Then agent 1^a (separately) meets agent 2^a and exchanges the banknote for good x_1^a . In subperiod 2, agent 1^b then redeems banknote on demand to its holder (either to agent 2^a on the equilibrium path or to agent 1^a off the equilibrium path). Again, as $\theta^b > 0$, agent 1^b finds the redemption of his note to be incentive compatible.

Therefore, depending on the severity of the matching and trading frictions, banknotes issued by sector b will arise naturally as a means of payment in sector a .

6 Appendix

6.1 Pareto Optimal Allocations

In this section, we characterize Pareto-optimal allocations in the case of no-temporal separation (i.e. ex-post Pareto optimal allocations, once types are known) and in the case of temporal separation (i.e. ex-post interim Pareto optimal allocations).

6.1.1 No Temporal Separation

With no-temporal separation, the set of Pareto optimal allocations is defined by the solutions to the following problem, considering that $\omega_1\gamma + \omega_2\gamma + \omega_3(1 - 2\gamma) = 1$.

$$\begin{aligned} & \max_{(x_1, x_2)} \quad \omega_1\gamma \left[U^1(x_1, x_2) + \frac{\beta}{1-\beta}\gamma S(x_1, x_2) \right] + \omega_2\gamma \left[U^2(x_2, x_1) + \frac{\beta}{1-\beta}\gamma S(x_1, x_2) \right] \\ & \quad + \omega_3(1 - 2\gamma) \frac{\beta\gamma}{1-\beta} S(x_1, x_2) \\ = & \max_{(x_1, x_2)} \quad \gamma \left[\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1) \right] + \frac{\beta\gamma}{1-\beta} S(x_1, x_2) \end{aligned}$$

The first order conditions are

$$\frac{\partial U^1(x_1, x_2)}{\partial x_1} = -\frac{\partial U^2(x_2, x_1)}{\partial x_1} \frac{\omega_2(1-\beta) + \beta}{\omega_1(1-\beta) + \beta}, \quad (44)$$

$$\frac{\partial U^2(x_2, x_1)}{\partial x_2} = -\frac{\partial U^1(x_1, x_2)}{\partial x_2} \frac{\omega_1(1-\beta) + \beta}{\omega_2(1-\beta) + \beta}. \quad (45)$$

From (44) and (45) we obtain the Pareto frontier as defined by

$$\frac{\partial U^1(x_1, x_2)}{\partial x_1} \frac{\partial U^2(x_2, x_1)}{\partial x_2} = \frac{\partial U^2(x_2, x_1)}{\partial x_1} \frac{\partial U^1(x_1, x_2)}{\partial x_2} \quad (46)$$

where $(x_1, x_2) \in \mathcal{I}$. The closed interval $\mathcal{I} \subseteq \mathbb{R}^2$ is defined from the minimum and maximum values of (x_1, x_2) from (44) and (45) which one gets when replacing $\omega_1 = 1/\gamma, \omega_2 = 0$, and $\omega_1 = 0, \omega_2 = 1/\gamma$.

Consider the case where $U^1(x_1, x_2) = u_1(x_1) - x_2$ and $U^2(x_2, x_1) = u_2(x_2) - x_1$.

Then the Pareto frontier is simply

$$u'_2(x_2) = \frac{1}{u'_1(x_1)} \quad (47)$$

where $u'_1(x_1) \in \left[\frac{\gamma\beta}{2(1-\beta)+\gamma\beta}, 1 + \frac{2(1-\beta)}{\gamma\beta} \right]$. Again, the limits on $u'_1(x_1)$ have been defined by replacing for both $\min \lambda_i = 0$ and $\max \lambda_i = 2/\gamma$ in (44). Finally, notice that, in this example, the Pareto frontier is concave, depending on the utility function and γ . From the equation defining the Pareto frontier we obtain that it is always decreasing

$$\frac{dx_2}{dx_1} = -\frac{u''_1(x_1)}{u''_2(x_2)} \frac{1}{[u'_1(x_1)]^2} < 0.$$

We also have the following result,

Lemma 1 *The Pareto frontier is concave if $u''_i(x_i) < 0$, and can be convex otherwise. The Pareto frontier shrinks as γ increases.*

Proof. For notational simplicity, we denote $u_1^{(n)}(x_1) = u_1^{(n)}$ for any degree n of the derivative. The curvature of the Pareto frontier is given by

$$\begin{aligned} \frac{(dx_2)^2}{d^2x_1} &= -\frac{u'''_1 u''_2 [u'_1]^2 - 2u''_1 u'_1 u''_2 [u'_1] - u'''_2 u''_1 [u'_1]^2 \frac{\partial x_2}{\partial x_1}}{(u''_2)^2 [u'_1]^4} \\ \frac{(u''_2)^2 [u'_1]^4}{u'_1 u''_2 u'_1} \frac{(dx_2)^2}{d^2x_1} &= 2u''_1 - u'_1 \left[\frac{u'''_1}{u''_1} - \frac{u''_2}{u'_1} \frac{\partial x_2}{\partial x_1} \right] \end{aligned}$$

So that the sign of $(dx_2)^2/d^2x_1$ is given by

$$\begin{aligned} \frac{(dx_2)^2}{d^2x_1} &= \text{sign} 2u_1'' - u_1' \left[\frac{u_1'''}{u_1''} - \frac{u_2'''}{u_2''} \frac{\partial x_2}{\partial x_1} \right] \\ &= \text{sign} 2u_1'' - u_1' \frac{u_1'''}{u_1''} - u_1' \frac{u_2'''}{[u_2'']^2} \frac{1}{u_1'} \end{aligned}$$

If $U^i < 0$, then the right hand side is negative. However, if $U^i > 0$, then the right hand side can be positive, for instance, if $u(x) = \sqrt{x}$ then the Pareto frontier is convex. To see that the Pareto frontier shrinks when γ increases, notice that $u_1'(x_1) \in \left[\frac{\gamma\beta}{2(1-\beta)+\gamma\beta}, 1 + \frac{2(1-\beta)}{\gamma\beta} \right]$. Hence, as γ increases, the upper bound for $u_1'(x_1)$ (the lower bound for x_1) decreases, while its lower bound (the upper bound for x_1) increases. ■

For example, for quadratic utility functions of the type $u(x) = ax - x^2$, the Pareto frontier is always concave.

Consider now the quasilinear case where $U^1(x_1, x_2) = x_1 - x_2$ and $U^2(x_1, x_2) = u(x_1) - x_2$. Then the objective function becomes

$$\gamma [\omega_1(x_1 - x_2) + \omega_2 u(x_2) - \omega_2 x_1] + \frac{\beta\gamma}{1-\beta} (u(x_2) - x_2)$$

The first order condition with respect to x_2 gives

$$u'(x_2) = \frac{\omega_1(1-\beta) + \beta}{\omega_2(1-\beta) + \beta}$$

However, if $\omega_1 = \omega_2$ then the optimal x_1 is on the positive real line, while if $\omega_1 > \omega_2$ then x_1 is infinity and if $\omega_2 > \omega_1$ then the optimal x_1 is zero (since production and consumption have to be positive).

6.1.2 Temporal Separation

With temporal separation, we also have to consider ex-post interim Pareto optimal allocations, in subperiod 2. These allocations are the solution to the following problem.

$$\max_{(x_1, x_2)} \omega_1 \gamma U^1(\bar{x}_1, x_2) + \omega_2 \gamma U^2(x_2, \bar{x}_1) + \frac{\beta\gamma}{1-\beta} S(x_1, x_2)$$

where \bar{x}_1 is the now sunk consumption/production in supperiod 1, and therefore cannot be maximized. At the optimal allocation $x_1 = \bar{x}_1$. The first order conditions are

$$\begin{aligned} \frac{\partial S(x_1, x_2)}{\partial x_1} &= 0 \\ \omega_1 \frac{\partial U^1(\bar{x}_1, x_2)}{\partial x_2} + \omega_2 \frac{\partial U^2(x_2, \bar{x}_1)}{\partial x_2} + \frac{\beta}{1-\beta} \frac{\partial S(x_1, x_2)}{\partial x_2} &= 0 \end{aligned}$$

Notice that if $\omega_1 = \omega_2$, then the solution is x^* .

Again consider the case where $U^1(x_1, x_2) = u_1(x_1) - x_2$ and $U^2(x_2, x_1) = u_2(x_2) - x_1$. Then the first order conditions imply

$$\begin{aligned} x_1 &= x_1^*, \text{ and} \\ u_2'(x_2) &= \frac{\omega_1(1-\beta) + \beta}{\omega_2(1-\beta) + \beta}. \end{aligned}$$

This is a vertical line in the (x_1, x_2) space.

Consider now the quasi-linear case where $U^1(x_1, x_2) = x_1 - x_2$ and $U^2(x_2, x_1) = u_2(x_2) - x_1$. The objective function is

$$\max_{(x_1, x_2)} -\omega_1 \gamma x_2 + \omega_2 \gamma u(x_2) + \frac{\beta \gamma}{1-\beta} (u(x_2) - x_2)$$

and x_1 is indeterminate, while x_2 solves

$$u_2'(x_2) = \frac{\omega_1(1-\beta) + \beta}{\omega_2(1-\beta) + \beta}.$$

6.2 Constrained Pareto Optimal Allocations

In this section, we characterize the set of constrained Pareto optimal allocations for the case of no-commitment and no-temporal separation for the case where $U^1(x_1, x_2) = u_1(x_1) - x_2$ and $U^2(x_2, x_1) = u_2(x_2) - x_1$. The constrained Pareto optimal allocations are the solutions to the following problem where the planner seeks to maximize the utility of traders and non-traders,

$$\begin{aligned} \max_{x_1, x_2} & \omega_1 \gamma [u_1(x_1) - x_2] + \omega_2 \gamma [u_2(x_2) - x_1] + \frac{\beta \gamma}{1-\beta} S(x_1, x_2) \\ & S(x_1, x_2) \geq 0, \\ & u_i(x_i) - x_j + \theta S(x_1, x_2) \geq 0. \end{aligned}$$

This is the problem of a planner at the start of a period, once types (non-traders, traders of type 1 and traders of type 2) have been revealed (as opposed to ex-ante, when agents do not know their types). Let η_1, η_2^i be the Lagrange multiplier on the two (generic) constraints. Then, the first order conditions are

$$\begin{aligned} 0 = & \gamma (\omega_i u'_i - \omega_j) + \frac{\theta}{\pi} (u'_i - 1) + \eta_1 (u'_i - 1) \\ & + \eta_2^i (u'_i + \theta (u'_i - 1)) + \eta_2^j (-1 + \theta (u'_i - 1)) \end{aligned}$$

Note that this first order condition can be simplified to

$$\gamma (\omega_j - \omega_i u'_i) - \eta_2^i u'_i + \eta_2^j = (u'_i - 1) \frac{\theta}{\pi} \left[1 + \frac{\pi \eta_1}{\theta \beta} + (\eta_2^i + \eta_2^j) \pi \right] \quad (48)$$

Dividing the first order conditions for type 1 agents using the first order condition for type 2 agents, we obtain

$$\frac{\gamma (\omega_2 - \omega_1 u'_1) - (\eta_2^1 u'_1 - \eta_2^2)}{\gamma (\omega_1 - \omega_2 u'_2) - (\eta_2^2 u'_2 - \eta_2^1)} = \frac{u'_1 - 1}{u'_2 - 1} \quad (49)$$

Recall that the Pareto set is defined by those allocations satisfying

$$u'_2 = \frac{1}{u'_1},$$

and (44). If, given λ , the corresponding Pareto optimal allocation is feasible, then replacing this expression in (49), it is easy to check that $\eta_2^1 + \eta_2^2 = 0$ is a solution, i.e. all *incentive* constraints are slack. Since (unconstrained) Pareto optimal allocation is unique, this is the only solution.

If however, given λ , the corresponding Pareto optimal allocation is not feasible, then $\eta_2^1 + \eta_2^2 > 0$, that is some *incentive* constraints bind and the constrained Pareto optimal allocation is on the frontier of the set of incentive feasible allocations. Notice in particular, that this is independent of the *participation* constraint binding. Also, suppose the set of incentive feasible allocations has positive Lebesgue-measure (it has a non-empty interior) then $\eta_2^1 \eta_2^2 = 0$. To the contrary, suppose there is a constrained Pareto optimal allocation (x_1, x_2) such that $\eta_2^1 \eta_2^2 > 0$. Then

$$u_i(x_i) - x_j + \theta S(x_1, x_2) = 0$$

for both $i = 1, 2$. Since the interior of the incentive feasible allocation set is non-empty, there is an incentive feasible allocation (\hat{x}_1, \hat{x}_2) such that

$$u_i(\hat{x}_i) - \hat{x}_j + \theta S(x_1, x_2) > 0$$

for at least one i . But this contradicts the fact that (x_1, x_2) was constrained Pareto optimal.

It remains to show that as ω_1 increases (or decreases) marginally, the constrained optimal allocation does not change. We know that either $\eta_2^1 > 0$, or $\eta_2^2 > 0$. Clearly, $\eta_2^1 > 0$ whenever ω_1 is low enough. Decreasing ω_1 , the left hand side of (48) increases. The only way to satisfy (48) is to either change the allocation or to increase η_2^1 (given $\eta_2^1 > 0$ increasing η_2^2 violates the result that $\eta_2^1 \eta_2^2 = 0$). Changing the allocation would contradict the fact that this allocation was optimal before the decrease in ω_1 (since the would-be new allocation was feasible), hence the allocation is unchanged and η_3^1 has to increase. A similar argument holds when ω_1 is high.

6.3 Proof of Theorem 2

The optimal allocation is defined by the solutions to the following problem, considering that $\omega_1 \gamma + \omega_2 \gamma + \omega_3 (1 - 2\gamma) = 1$,

$$\max_{(x_1, x_2)} \mathcal{W}(x_1, x_2) = \max_{(x_1, x_2)} \gamma [\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1)] + \frac{\theta}{\pi} S(x_1, x_2)$$

The first order conditions are (44) and (45) which we rewrite here for convenience

$$\begin{aligned} \gamma \left[\omega_1 \frac{\partial U^1}{\partial x_1} + \omega_2 \frac{\partial U^2}{\partial x_1} \right] + \frac{\theta}{\pi} \frac{\partial S}{\partial x_1} &= 0 \\ \gamma \left[\omega_1 \frac{\partial U^1}{\partial x_2} + \omega_2 \frac{\partial U^2}{\partial x_2} \right] + \frac{\theta}{\pi} \frac{\partial S}{\partial x_2} &= 0 \end{aligned}$$

and replacing the expression for S and θ , we get

$$\begin{aligned} \frac{\partial U^1(x_1, x_2)}{\partial x_1} &= - \frac{\partial U^2(x_2, x_1)}{\partial x_1} \frac{\omega_2 (1 - \beta) + \beta}{\omega_1 (1 - \beta) + \beta}, \\ \frac{\partial U^2(x_2, x_1)}{\partial x_2} &= - \frac{\partial U^1(x_1, x_2)}{\partial x_2} \frac{\omega_1 (1 - \beta) + \beta}{\omega_2 (1 - \beta) + \beta}. \end{aligned}$$

Notice that the Pareto optimal allocations as defined by (44) and (45) do not depend on the trading intensity γ^i in any sector. Therefore, the optimal allocation is the same across sectors. Then we obtain the Pareto frontier as defined by

$$\frac{\partial U^1(x_1, x_2)}{\partial x_1} \frac{\partial U^2(x_2, x_1)}{\partial x_2} = \frac{\partial U^2(x_2, x_1)}{\partial x_1} \frac{\partial U^1(x_1, x_2)}{\partial x_2} \quad (50)$$

where $(x_1, x_2) \in \mathcal{I}$.²⁴

Later we will need to know that the Pareto frontier is monotone decreasing. Totally differentiating (50) we get

$$\frac{dx_2}{dx_1} = \frac{\frac{\partial^2 U^1}{(\partial x_1)^2} \frac{\partial U^2}{\partial x_2} + \frac{\partial U^1}{\partial x_1} \frac{\partial^2 U^2}{\partial x_2 \partial x_1} - \left(\frac{\partial^2 U^2}{(\partial x_1)^2} \frac{\partial U^1}{\partial x_2} + \frac{\partial U^2}{\partial x_1} \frac{\partial^2 U^1}{\partial x_2 \partial x_1} \right)}{\frac{\partial^2 U^2}{\partial x_1 \partial x_2} \frac{\partial U^1}{\partial x_2} + \frac{\partial U^2}{\partial x_1} \frac{\partial^2 U^1}{(\partial x_2)^2} - \left(\frac{\partial^2 U^1}{\partial x_1 \partial x_2} \frac{\partial U^2}{\partial x_2} + \frac{\partial U^1}{\partial x_1} \frac{\partial^2 U^2}{(\partial x_2)^2} \right)} < 0$$

where the negative sign comes from the usual assumptions that for all (x_1, x_2)

$$\begin{aligned} \frac{\partial U^i(x_i, x_j)}{\partial x_j} < 0; \quad \frac{\partial U^i(x_i, x_j)}{\partial x_i} > 0; \quad \frac{\partial^2 U^i(x_i, x_j)}{(\partial x_i)^2} \leq 0; \quad \frac{\partial^2 U^i(x_i, x_j)}{\partial x_i \partial x_j} \leq 0; \text{ and} \\ -x_i \frac{\partial^2 U^i(x_i, x_j)}{(\partial x_i)^2} / \frac{\partial U^i(x_i, x_j)}{\partial x_i} \geq -x_i \frac{\partial^2 U^j(x_j, x_i)}{(\partial x_i)^2} / \frac{\partial U^j(x_j, x_i)}{\partial x_i} \end{aligned}$$

(\hat{x}_1, \hat{x}_2) solves the following problem

$$\begin{aligned} \max_{(x_1, x_2)} \quad & \gamma [\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1)] + \frac{\theta}{\pi} S(x_1, x_2) \\ \text{s.t.} \quad & \\ & U^1(x_1, x_2) + \theta S(x_1, x_2) \geq 0 \\ & U^2(x_2, x_1) + \theta S(x_1, x_2) \geq 0 \\ & -\lambda x_2 + \theta S(x_1, x_2) \geq 0 \end{aligned}$$

Denote the Lagrange multiplier on the first to last constraints as ν , η and ρ respectively. Then we have the following result

Claim 6 *If $\rho = 0$ then (\hat{x}_1, \hat{x}_2) lies on the Pareto frontier.*

²⁴The closed interval $\mathcal{I} \subseteq \mathbb{R}^2$ is defined from the minimum and maximum values of (x_1, x_2) from (44) and (45) which one gets when replacing $\lambda_1 = 1/\gamma, \lambda_2 = 0$, and $\lambda_1 = 0, \lambda_2 = 1/\gamma$.

The first order conditions to the problem above are

$$\begin{aligned} \gamma\lambda_1 \frac{\partial U^1}{\partial x_1} + \gamma\lambda_2 \frac{\partial U^2}{\partial x_1} + \frac{\theta}{\pi} \frac{\partial S}{\partial x_1} + \nu \left(\frac{\partial U^1}{\partial x_1} + \theta \frac{\partial S}{\partial x_1} \right) + \eta \left(\frac{\partial U^2}{\partial x_1} + \theta \frac{\partial S}{\partial x_1} \right) + \rho \theta \frac{\partial S}{\partial x_1} &= 0 \\ \gamma\lambda_1 \frac{\partial U^1}{\partial x_2} + \gamma\lambda_2 \frac{\partial U^2}{\partial x_2} + \frac{\theta}{\pi} \frac{\partial S}{\partial x_2} + \nu \left(\frac{\partial U^1}{\partial x_2} + \theta \frac{\partial S}{\partial x_2} \right) + \eta \left(\frac{\partial U^2}{\partial x_2} + \theta \frac{\partial S}{\partial x_2} \right) + \rho \left(-\lambda + \theta \frac{\partial S}{\partial x_2} \right) &= 0 \end{aligned}$$

Suppose $\eta \geq 0$, and $\nu \geq 0$ while $\rho = 0$. Then the FOCs become

$$\begin{aligned} (\gamma\omega_1 + \nu) \frac{\partial U^1}{\partial x_1} + (\gamma\omega_2 + \eta) \frac{\partial U^2}{\partial x_1} + \frac{\theta}{\pi} \frac{\partial S}{\partial x_1} + (\nu + \eta) \theta \frac{\partial S}{\partial x_1} &= 0 \\ (\gamma\omega_1 + \nu) \frac{\partial U^1}{\partial x_2} + (\gamma\omega_2 + \eta) \frac{\partial U^2}{\partial x_2} + \frac{\theta}{\pi} \frac{\partial S}{\partial x_2} + (\nu + \eta) \theta \frac{\partial S}{\partial x_2} &= 0 \end{aligned}$$

and using the fact that $\frac{\partial S}{\partial x_i} = \frac{\partial U^1}{\partial x_i} + \frac{\partial U^2}{\partial x_i}$ and arranging terms, we obtain

$$\begin{aligned} \left(\gamma\omega_1 + \nu + \frac{\theta}{\pi} + (\nu + \eta) \theta \right) \frac{\partial U^1}{\partial x_1} + \left(\gamma\omega_2 + \eta + \frac{\theta}{\pi} + (\nu + \eta) \theta \right) \frac{\partial U^2}{\partial x_1} &= 0 \\ \left(\gamma\omega_1 + \nu + \frac{\theta}{\pi} + (\nu + \eta) \theta \right) \frac{\partial U^1}{\partial x_2} + \left(\gamma\omega_2 + \eta + \frac{\theta}{\pi} + (\nu + \eta) \theta \right) \frac{\partial U^2}{\partial x_2} &= 0 \end{aligned}$$

Therefore, we get

$$\frac{\partial U^1}{\partial x_1} / \frac{\partial U^1}{\partial x_2} = \frac{\partial U^2}{\partial x_1} / \frac{\partial U^2}{\partial x_2}$$

so that, if $\rho = 0$, then the solution lies on the Pareto frontier. Looking at the original first order conditions, it is easy to see that if $\rho > 0$, the conditions are no longer symmetric and therefore the solution does not necessarily lie on the Pareto frontier. This proves the claim.

We now introduce some notation for the curves that define the frontiers of \mathcal{F}_0^i .

Let

$$\begin{aligned} C_1(\theta^i) &\equiv \{(x_1, x_2) : U^1(x_1, x_2) + \theta^i S(x_1, x_2) = 0\} \\ C_2(\theta^i) &\equiv \{(x_1, x_2) : U^2(x_2, x_1) + \theta^i S(x_1, x_2) = 0\} \\ C_r(\theta^i) &\equiv \{(x_1, x_2) : -\lambda x_2 + \theta^i S(x_1, x_2) = 0\} \end{aligned}$$

Notice that if (x_1, x_2) is in any $C_k(\theta^i)$, $k = 1, 2, r$, then it is not necessarily in \mathcal{F}_0^i . We will be interested the properties of those curve for points that are also feasible, ie. those points in $C_k(\theta^i) \cap \mathcal{F}_0^i$. Using Inada conditions and assuming a single crossing

property, we obtain that the curve $C_1(\theta^i) \cap \mathcal{F}_0^i$ lies above (in the (x_1, x_2) -space) the curve $C_2(\theta^i) \cap \mathcal{F}_0^i$, in the sense that for all $(x_1, x_2) \in C_1(\theta^i) \cap \mathcal{F}_0^i$, there is $x'_2 \leq x_2$ such that $(x_1, x'_2) \in C_2(\theta^i) \cap \mathcal{F}_0^i$. Notice that the origin $(x_1, x_2) = (0, 0)$ belongs to all three curves and is therefore also in \mathcal{F}_0^i .

Claim 7 *Suppose $(x_1^*, x_2^*) \in \mathcal{F}_0^a$. Then deposits are not Pareto essential.*

Since $\theta^b > \theta^a$ it is easy to show that $\mathcal{F}_0^a \subset \mathcal{F}_0^b$. Since $(x_1^*, x_2^*) \in \mathcal{F}_0^a$, then $(x_1^*, x_2^*) \in \mathcal{F}_0^b$. Therefore the solution is $\hat{x}^a = \hat{x}^b = x^*$. Hence the planner cannot do better in any sector using deposits.

Claim 8 *Suppose $(x_1^*, x_2^*) \in \mathcal{F}_0^b / \mathcal{F}_0^a$, then only deposits in sector b are Pareto essential.*

Since $(x_1^*, x_2^*) \in \mathcal{F}_0^b / \mathcal{F}_0^a$, we know that $\hat{x}^b = x^*$, while $\hat{x}^a \in C_k(\theta^a)$ for some $k = 1, 2, r$. If this is not the case then $\hat{x}^a = x^*$, which violates the fact that x^* is not feasible in sector a . Since deposits only relaxes the repayment constraint, deposits in sector b can improve upon \hat{x}_a in sector a whenever $\hat{x}^a \in C_r(\theta^a)$. Otherwise, at the margin, deposits have no role.

Claim 9 *Suppose $(x_1^*, x_2^*) \notin \mathcal{F}_0^b$, then only deposits in sector b are Pareto essential.*

To prove the claim, it is enough to show that if $\hat{x}^b \in C_r(\theta^b)$ then $\hat{x}^a \in C_r(\theta^a)$, or equivalently, if $\hat{x}^a \in \mathcal{F}_0^a \setminus C_r(\theta^a)$ then $\hat{x}^b \in \mathcal{F}_0^b \setminus C_r(\theta^b)$.

Suppose by way of contradiction that $\hat{x}^a \notin C_r(\theta^a)$ while $\hat{x}^b \in C_r(\theta^b)$. We want to show that there is a *feasible* allocation x' in sector b such that $\mathcal{W}^b(x') > \mathcal{W}^b(\hat{x}^b)$.

First, notice as $(x_1^*, x_2^*) \notin \mathcal{F}_0^b$ then $(x_1^*, x_2^*) \notin \mathcal{F}_0^a$. Since $\hat{x}^a \notin C_r(\theta^a)$ and $(x_1^*, x_2^*) \notin \mathcal{F}_0^a$, it must be that either $\hat{x}^a \in C_1(\theta^a)$ or $\hat{x}^a \in C_2(\theta^a)$ (i.e. either $\nu^a > 0$ or $\eta^a > 0$). In any case, we showed earlier that if $\nu^a + \eta^a > 0$ and $\rho^a = 0$, then \hat{x}^a is on the Pareto frontier. In particular, this means that the Pareto frontier intersects either $C_1(\theta^a)$ or $C_2(\theta^a)$.

We will need the following result, which states that the intersection between the dynamic participation constraint for type 1 and the repayment constraint shifts to the north-east as θ increases when $C_r(\theta^i)$ crosses $C_1(\theta^i)$ from above. Otherwise, the intersection moves to the south-west as θ increases. We assume that $C_r(\theta^i)$ and $C_1(\theta^i)$ cross only once away from the origin.

Claim 10 *Let $\check{x}^i = C_1(\theta^i) \cap C_r(\theta^i)$. Suppose $C_r(\theta^i)$ crosses $C_1(\theta^i)$ from above. Then $\check{x}^b > \check{x}^a$ in the sense that $\check{x}_1^b \geq \check{x}_1^a$ and $\check{x}_2^b \geq \check{x}_2^a$. Suppose $C_r(\theta^i)$ crosses $C_1(\theta^i)$ from below, then $\check{x}^b < \check{x}^a$ in the sense that $\check{x}_1^b \leq \check{x}_1^a$ and $\check{x}_2^b \leq \check{x}_2^a$.*

By definition, \check{x}^i satisfies

$$\begin{aligned} U^1(\check{x}_1^i, \check{x}_2^i) + \theta^i S(\check{x}_1^i, \check{x}_2^i) &= 0 \\ -\lambda \check{x}_2^i + \theta^i S(\check{x}_1^i, \check{x}_2^i) &= 0 \end{aligned}$$

We assume here it is unique. First notice that \check{x}_1^i is such that $U^1(\check{x}_1^i, \check{x}_2^i) = -\lambda \check{x}_2^i$ for $i = a, b$. Since $U^1(x_1, x_2) - U^1(x_1, 0) \leq -\lambda x_2$, for all (x_1, x_2) we easily get for all x_1

$$\frac{\partial U^1(x_1, x_2)}{\partial x_2} \leq -\lambda.$$

To prove the claim, it is enough to show that as θ increases, $C_r(\theta)$ shifts by more than $C_1(\theta)$. In other words, keeping \check{x}_1^a constant, we need to show that the solution \check{x}_2 to

$$U^1(\check{x}_1^a, \check{x}_2) + (\theta^a + \varepsilon) S(\check{x}_1^a, \check{x}_2) = 0 \tag{51}$$

is smaller than the solution x_2' to

$$-\lambda x_2' + (\theta^a + \varepsilon) S(\check{x}_1^a, x_2') = 0 \tag{52}$$

for arbitrarily small ε . Equivalently, since $\check{x}_2 = x_2' = \check{x}_2^a$ when $\varepsilon = 0$, we need to show that

$$\frac{d\check{x}_2}{d\theta^a} < \frac{dx_2'}{d\theta^a}.$$

Setting $\varepsilon = 0$ and totally differentiating (51) while keeping \check{x}_1^a constant, we obtain

$$\frac{d\check{x}_2}{d\theta^a} = \frac{-S}{\frac{\partial U^1}{\partial \check{x}_2} + \theta^a \frac{\partial S}{\partial \check{x}_2}} = \frac{\lambda \check{x}_2^a / \theta^a}{-\frac{\partial U^1}{\partial \check{x}_2} - \theta^a \frac{\partial S}{\partial \check{x}_2}}$$

where the last equality follows from the fact that at θ^a , $\check{x}_2 = \check{x}_2^a$ so that $-\lambda\check{x}_2^a + \theta^a S(\check{x}_1^a, \check{x}_2^a) = 0$. Notice that the denominator is positive, by definition of $C_1(\theta^a)$ (for any allocation on $C_1(\theta^a)$, we have $U^1 + \theta^a S = 0$ and any allocation above $C_1(\theta^a)$ satisfy $U^1 + \theta^a S < 0$, hence starting from an allocation on $C_1(\theta^a)$ and increasing x_2 while leaving x_1 constant brings us above $C_1(\theta^a)$ so that $\frac{\partial U^1}{\partial x_2} + \theta^a \frac{\partial S}{\partial x_2} < 0$.) As a consequence, $\frac{d\check{x}_2}{d\theta^a} > 0$. Now, totally differentiating (52), we obtain

$$\frac{dx'_2}{d\theta^a} = \frac{-S}{-\lambda + \theta^a \frac{\partial S}{\partial x_2}} = \frac{\lambda\check{x}_2^a/\theta^a}{\lambda - \theta^a \frac{\partial S}{\partial \check{x}_2}}$$

where again the denominator is positive. Since $\lambda \leq -\frac{\partial U^1}{\partial x_2}$, we get that $\frac{dx'_2}{d\theta^a} > \frac{d\check{x}_2}{d\theta^a}$, which proves the claim.

Using a similar argument, we can also show:

Claim 11 *Let $\xi^i = C_2(\theta^i) \cap C_r(\theta^i)$, then $\xi^b > \xi^a$ in the sense that $\xi_1^b \geq \xi_1^a$ and $\xi_2^b \geq \xi_2^a$.*

Notice that at $(0, 0)$, the slope along $C_r(\theta^i)$ is larger than the slope of $C_2(\theta^i)$. Hence, if ξ^i exists, it must be that $C_r(\theta^i)$ crosses $C_2(\theta^i)$ from above. Then the proof is the same as before except that x_2 should now be held constant. It can then easily be shown that $dx'_1/d\theta^a > d\check{x}_1/d\theta^a$, so that $C_r(\theta^a)$ shifts again more than $C_2(\theta^a)$. [Here is the proof for the sake of completeness: keeping \check{x}_2^a constant, we need to show that the solution \check{x}_1 to

$$U^2(\xi_2^a, \check{x}_1) + (\theta^a + \varepsilon) S(\check{x}_1, \xi_2^a) = 0 \tag{53}$$

is smaller than the solution x'_1 to

$$-\lambda\xi_2^a + (\theta^a + \varepsilon) S(x'_1, \xi_2^a) = 0 \tag{54}$$

for arbitrarily small ε . Equivalently, since $\check{x}_1 = x'_1 = \check{x}_1^a$ when $\varepsilon = 0$, we need to show that

$$\frac{d\check{x}_1}{d\theta^a} < \frac{dx'_1}{d\theta^a}.$$

Setting $\varepsilon = 0$ and totally differentiating (53) while keeping \check{x}_2^a constant, we obtain

$$\frac{d\check{x}_1}{d\theta^a} = \frac{-S}{\frac{\partial U^2}{\partial \check{x}_1} + \theta^a \frac{\partial S}{\partial \check{x}_1}} = \frac{\lambda \xi_2^a / \theta^a}{-\frac{\partial U^2}{\partial \check{x}_1} - \theta^a \frac{\partial S}{\partial \check{x}_1}}$$

where the last equality follows from the fact that at θ^a , $\check{x}_1 = \xi_1^a$ so that $-\lambda \xi_2^a + \theta^a S(\xi_1^a, \xi_2^a) = 0$. Notice that the denominator is positive, by definition of $C_2(\theta^a)$ (for any allocation on $C_2(\theta^a)$, we have $U^2 + \theta^a S = 0$ and any allocation below $C_2(\theta^a)$ satisfy $U^2 + \theta^a S < 0$, hence starting from an allocation on $C_2(\theta^a)$ and increasing x_1 while leaving x_2 constant brings us below $C_2(\theta^a)$ so that $\frac{\partial U^2}{\partial x_1} + \theta^a \frac{\partial S}{\partial x_1} < 0$.) As a consequence, $\frac{d\check{x}_2}{d\theta^a} > 0$. Now, totally differentiating (52), we obtain

$$\frac{dx_1'}{d\theta^a} = \frac{-S}{\theta^a \frac{\partial S}{\partial \check{x}_1}} = \frac{\lambda \xi_2^a / \theta^a}{-\theta^a \frac{\partial S}{\partial \check{x}_1}}$$

where again the denominator is positive. Since $-\frac{\partial U^2}{\partial \check{x}_1} \geq 0$, we get that $\frac{dx_1'}{d\theta^a} > \frac{d\check{x}_1}{d\theta^a}$, which proves the claim.]

Claim 12 *If the Pareto frontier intersects $C_k(\theta^a)$ for $k = 1, 2$, then it also intersects $C_k(\theta^b)$.*

Recall that $\mathcal{F}_0^a \subset \mathcal{F}_0^b$ and the Pareto frontier intersects $C_k(\theta^a)$ for $k = 1$ or 2 , so that if the Pareto frontier intersects \mathcal{F}_0^a , then it also intersects \mathcal{F}_0^b . The result then follows from claims (10) and (11) together with the fact that the Pareto frontier is a monotone decreasing function.

Now notice that moving along the Pareto frontier toward x^* is always welfare improving. More precisely, suppose at x^* neither the dynamic participation constraints for type 1 or type 2 is satisfied. In particular, x^* is not feasible in sector i , for any i . Then the solution to

$$\max_{(x_1, x_2)} \gamma [\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1)] + \frac{\theta}{\pi} S(x_1, x_2)$$

s.t.

$$U^1(x_1, x_2) + \theta S(x_1, x_2) \geq 0$$

$$U^2(x_2, x_1) + \theta S(x_1, x_2) \geq 0$$

will lie on the intersection between the Pareto frontier and the curve $C_k(\theta^i)$ which is closest to x^* . Notice that the problem above ignores the repayment constraint. Now suppose $\hat{x}^a \in C_k(\theta^a)$ for $k = 1$ or 2 . Then \hat{x}^a is the solution to

$$\begin{aligned} & \max_{(x_1, x_2)} \gamma [\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1)] + \frac{\theta^a}{\pi} S(x_1, x_2) \\ & s.t. \\ & U^k(x_1, x_2) + \theta^a S(x_1, x_2) \geq 0 \end{aligned}$$

and lies on the Pareto frontier. Now, consider the solution \hat{x}^b to

$$\begin{aligned} & \max_{(x_1, x_2)} \gamma [\omega_1 U^1(x_1, x_2) + \omega_2 U^2(x_2, x_1)] + \frac{\theta^b}{\pi} S(x_1, x_2) \\ & s.t. \\ & U^k(x_1, x_2) + \theta^b S(x_1, x_2) \geq 0 \end{aligned}$$

\hat{x}^b will be preferred to \hat{x}^a as it is the solution to the problem that ignores the constraint that is binding for \hat{x}^a . In addition, \hat{x}^b will be the point of the intersection between the Pareto frontier and $C_k(\theta^b)$. We know this point exists and is in \mathcal{F}_0^b by the last claim above. This contradicts the fact that \hat{x}^a is the solution to the planner's problem. This proves (9) and the Theorem.

6.4 Example

Here we show that the solution to (35) features transfers and deposits only from sector b . More precisely, in the case where $U_1(x_1, x_2) = -x_2$ and $U_2(x_2, x_1) = u(x_2)$, we show

Proposition 1 *Suppose $\theta^b > \theta^a$ and $x^* \notin \mathcal{F}_0^a$. Then the optimal allocation is such that*

$$\frac{\theta^a}{1 + \theta^a} u(x_2^a) - x_2^a < 0 \text{ and } \frac{\theta^b}{1 + \theta^b} u(x_2^b) - x_2^b > 0.$$

Proof. Suppose $\theta^a < \theta^b$ and $\bar{x}_2^b > x^*$, so that x^* is feasible in the b sector. If $x_2^a \geq x^*$ is feasible, then the solution is necessarily x^* ($x_2^a > x^*$ and $x_2^b > x^*$ is not

optimal, and $x_2^a > x^*$, $x_2^b < x^*$ violates the result that $\theta^b > \theta^a$, then $x_2^b > x_2^a$). Therefore, $x_2^a \leq x^*$. Suppose now $x_2^b > x^*$ is optimal. Then there is $\tilde{x}_2^b < x_2^b$ such that $u(\tilde{x}_2^b) - \tilde{x}_2^b > u(x_2^b) - x_2^b$ and $\frac{\theta^b}{1+\theta^b}u(\tilde{x}_2^b) - \tilde{x}_2^b > \frac{\theta^b}{1+\theta^b}u(x_2^b) - x_2^b$ (since $\tilde{x}_2^b < x^*$) The planner can therefore pick the allocation \tilde{x}_2^b instead for the b -sector and increase slightly transfer to the a -sector. This contradicts that $x_2^b > x^*$ is optimal. In particular, any allocation such that $x_2^b > \bar{x}_2^b$ where bankers are in the a -sector is suboptimal. Hence, we get that if $\theta^a < \theta^b$ and $\bar{x}_2^b > x^*$, then the optimal solution (x_2^a, x_2^b) satisfies $x_2^a \leq x_2^b \leq x^*$, and $\frac{\theta^b}{1+\theta^b}u(x_2^b) - x_2^b > 0$. Therefore, bankers are in the b -sector.

Suppose now $\theta^a < \theta^b$ and $\bar{x}_2^b < x^*$, so that x^* is not feasible in the b -sector. Then $\bar{x}_2^a < \bar{x}_2^b < x^*$, so that x^* is not feasible in the a -sector either. Then we know that the constraint binds, as x^* will not be feasible even with transfers and deposits. To show that an allocation (x_2^a, x_2^b) such that $x_2^a < \bar{x}_2^a < \bar{x}_2^b < x_2^b$ is not optimal, we start from the case where $\theta^a = \theta^b$, and increase θ^b slightly while leaving θ^a constant. We show that the solution then can't be in the range $x_2^a < \bar{x}_2^a < \bar{x}_2^b < x_2^b$.

When $\theta^a = \theta^b$, we know that the solution is $x_2^a = x_2^b = \bar{x}_2^a = \bar{x}_2^b$, and the slope of the constraint and of the indifference curve are equal. Increasing θ^b moves \bar{x}_2^b up, while leaving \bar{x}_2^a constant. Therefore, the change in the slope of the indifference curve at $(\bar{x}_2^a, \bar{x}_2^b)$ is

$$\begin{aligned} & -\frac{\gamma^a u''(\bar{x}_2^a) [u'(\bar{x}_2^b) - 1] \frac{d\bar{x}_2^a}{d\theta^b} - u''(\bar{x}_2^b) [u'(\bar{x}_2^a) - 1] \frac{d\bar{x}_2^b}{d\theta^b}}{\gamma^b [u'(\bar{x}_2^b) - 1]^2} \\ & = \frac{\gamma^a u''(\bar{x}_2^b) [u'(\bar{x}_2^a) - 1] \frac{d\bar{x}_2^b}{d\theta^b}}{\gamma^b [u'(\bar{x}_2^b) - 1]^2} < 0 \end{aligned}$$

since we know that $\frac{d\bar{x}_2^b}{d\theta^b} > 0$ and $u''(\cdot) < 0$. Therefore, the slope of the indifference curve decreases and as it was already negative, it becomes *more* steep at $(\bar{x}_2^a, \bar{x}_2^b)$.

Similarly, the change in the slope of the constraint curve at $(\bar{x}_2^a, \bar{x}_2^b)$ is

$$\begin{aligned} & \frac{\gamma^a \frac{\theta^a}{1+\theta^a} u''(\bar{x}_2^a) \left[\frac{\theta^b}{1+\theta^b} u'(\bar{x}_2^b) - 1 \right] \frac{d\bar{x}_2^a}{d\theta^b} - \frac{\theta^b}{1+\theta^b} u''(\bar{x}_2^b) \left[\frac{\theta^a}{1+\theta^a} u'(\bar{x}_2^a) - 1 \right] \frac{d\bar{x}_2^b}{d\theta^b}}{\gamma^b \left[\frac{\theta^b}{1+\theta^b} u'(\bar{x}_2^b) - 1 \right]^2} \\ &= \frac{\gamma^a}{\gamma^b} \frac{\theta^b}{1+\theta^b} \frac{u''(\bar{x}_2^b) \left[\frac{\theta^a}{1+\theta^a} u'(\bar{x}_2^a) - 1 \right] \frac{d\bar{x}_2^b}{d\theta^b}}{\left[\frac{\theta^b}{1+\theta^b} u'(\bar{x}_2^b) - 1 \right]^2} > 0 \end{aligned}$$

This is positive as $\frac{d\bar{x}_2^b}{d\theta^b} > 0$, $u''(\cdot) < 0$ and $\frac{\theta^a}{1+\theta^a} u'(\bar{x}_2^a) < 1$ by definition of \bar{x}_2^a . Therefore, the slope of the constraint curve increases, and as it was negative, this means it becomes *less* steep at $(\bar{x}_2^a, \bar{x}_2^b)$.

As the indifference curve is more steep and the constraint curve is less steep at $(\bar{x}_2^a, \bar{x}_2^b)$, this means that a tangency can only occur in the region $\bar{x}_2^a < x_2^a < x_2^b < \bar{x}_2^b$, and not in the region where $x_2^a < \bar{x}_2^a < \bar{x}_2^b < x_2^b$. In words, as θ^b becomes larger than θ^a , the solution to the planner's problem implies that bankers are in sector b . Notice that the proof is holding γ^b constant. However increasing γ^b from $\gamma^b = \gamma^a$ will lead the same result as it also increases θ^b . The only difference is in the γ^a/γ^b term, but this will not matter as at the original $(\bar{x}_2^a, \bar{x}_2^b)$ the two slopes are the same and we therefore only add up a constant. Therefore the result that one curve becomes steeper than the other is still there. ■

6.5 Positive Rate of Return on Storage

The planner's problem with no interaction between sectors is given by (41). The first best solution (i.e. the solution ignoring incentive constraint) is given by x_2^{*i} solving

$$u'(x_2^{*,i}) = \frac{1}{1 + \rho^i} \quad (55)$$

Therefore $x_2^{*,a} > x_2^{*,b}$, since $\rho > 0$. Now, we denote by \underline{x}_2^a the level of x_2^a that satisfies the repayment constraint (43) as an equality. Then \underline{x}_2^a and \underline{x}_2^b are defined by, respectively

$$\frac{u(\underline{x}_2^a)}{\underline{x}_2^a} = \frac{1}{\theta^a} + \frac{1}{1 + \rho} \quad (56)$$

$$\frac{u(x_2^b)}{x_2^b} = \frac{1}{\theta^b} + 1 \quad (57)$$

Because of concavity $u(x_2^i)/x_2^i$ is decreasing x_2^i and therefore we have

$$\underline{x}_2^b \geq \underline{x}_2^a \quad \text{iff} \quad \theta^b - \theta^a \geq \theta^a \theta^b \frac{\rho}{1 + \rho} \quad (58)$$

We first compute the level of market connection $\bar{\theta}^i$ below which the repayment constraint binds in sector i . This is the level of θ such that the repayment constraint is satisfied with equality at $x_2^{*,i}$. Therefore $\bar{\theta}^a$ is such that

$$\frac{u(x_2^{*,a})}{x_2^{*,a}} = \frac{1}{\bar{\theta}^a} + \frac{1}{1 + \rho}$$

while $\bar{\theta}^b$ is such that

$$\frac{u(x_2^{*,b})}{x_2^{*,b}} = \frac{1}{\bar{\theta}^b} + 1$$

We have to consider depending on whether the repayment constraints bind at the planner's solution when there is no deposit or transfer.

6.5.1 $\theta^i \geq \bar{\theta}^i$ for $i = a, b$

In this case the repayment constraint does not bind in any sector at the solution to (41). Deposits are not Pareto essential since both sectors are already at $x_2^{*,i}$.

6.5.2 $\theta^a \geq \bar{\theta}^a$ and $\theta^b < \bar{\theta}^b$

In this case, the repayment constraints bind in sector b but not in sector a at the solution to (41). We have the following result.

Claim 13 *Deposits in sector a are Pareto essential if*

$$\theta^a \geq \bar{\theta}^a \text{ and } \theta^b < \bar{\theta}^b$$

Without deposits the production constraint and the repayment constraint in sector b coincide. They are both given by

$$-x_2^b + \theta^b [u(x_2^b) - x_2^b] = 0$$

Suppose agents in sector b deposit an amount \tilde{d} with sector a . Given allocation x_2^b , agents in sector b have to produce \tilde{y}^b such that

$$\begin{aligned} x_2^b &= \tilde{y}^b - \tilde{d} + \tilde{d}(1 + \rho), \text{ or} \\ \tilde{y}^b &= x_2^b - \tilde{d}\rho \end{aligned} \tag{59}$$

Then the repayment constraint in sector b becomes

$$-\left(\tilde{y}^b - \tilde{d}\right) + \theta^b [u(x_2^b) - \tilde{y}^b] > 0$$

while the production constraint remains

$$-\tilde{y}^b + \theta^b [u(x_2^b) - \tilde{y}^b] \geq 0.$$

Therefore, with deposits $\tilde{d} > 0$ and no transfers, the production constraint becomes the relevant constraint. Replacing (59) in the production constraint in sector b we obtain

$$-\left(x_2^b - \tilde{d}\rho\right) + \theta^b \left[u(x_2^b) - \left(x_2^b - \tilde{d}\rho\right)\right] \geq 0.$$

Hence, deposits relax the constraint set. As a consequence, if $\theta^b < \bar{\theta}^b$ while $\theta^a \geq \bar{\theta}^a$ such that the repayment constraint binds in sector b but not in sector a , then deposits in sector a are Pareto essential.

6.5.3 $\theta^a < \bar{\theta}^a$ and $\theta^b < \bar{\theta}^b$ (sector a takes deposits)

In this case, both repayment constraints bind at the solution to (41). We first analyze the case when sector a takes deposits. We have the following result

Claim 14 *Deposits in sector a are Pareto essential if*

$$\theta^a < \bar{\theta}^a, \theta^b < \bar{\theta}^b \text{ and } \rho\theta^a > 1.$$

When $\theta^a < \bar{\theta}^a$, the solution to (41) in sector a is \underline{x}_2^a , and the repayment constraint binds. Therefore, if agents in sector a receive a positive amount of deposits they will default. Deposits in sector a are incentive compatible only if agents in sector b also

make a transfer t to agents in sector a . We assume that transfers are made at the beginning of the period, stored and then consumed. Then, the repayment constraint in sector a with transfer t and taking deposits \tilde{d} is

$$-x_2^a - \tilde{d}(1 + \rho) + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} + t(1 + \rho) \right] \geq 0.$$

Without loss of generality, we assume that the planner leaves the allocation in sector a at the solution to (41), \underline{x}_2^a . By definition of \underline{x}_2^a , the repayment constraint at \underline{x}_2^a becomes $t \geq \tilde{d}/\theta^a$, so that the minimum transfer necessary to make \tilde{d} incentive compatible is

$$t = \tilde{d}/\theta^a. \tag{60}$$

Let us now consider the incentive constraints in sector b . The production constraint is

$$-x_2^b + \tilde{d}\rho - t + \theta^b \left[u(x_2^b) - x_2^b + \tilde{d}\rho - t \right] \geq 0, \tag{61}$$

while the repayment constraint is

$$-x_2^b + \tilde{d}(1 + \rho) + \theta^b \left[u(x_2^b) - x_2^b + \tilde{d}\rho - t \right] \geq 0$$

Clearly the production constraint is the relevant one, and deposits relax the constraint set, while transfers tighten it. Using (60) in (61) we obtain the production constraint in sector b

$$-x_2^b + \tilde{d}\rho - \tilde{d}/\theta^a + \theta^b \left[u(x_2^b) - x_2^b + \tilde{d}\rho - \tilde{d}/\theta^a \right] \geq 0. \tag{62}$$

Therefore, deposits relax the production constraint of sector b if and only if

$$\rho > \frac{1}{\theta^a}$$

Finally, we check that increasing deposits actually increase welfare in sector b given agents in sector b makes a transfer to agents in sector a . From the production constraint (61), we get the minimum \tilde{d} necessary to implement any $x_2^b > \underline{x}_2^a$,

$$\tilde{d} \left(\rho - \frac{1}{\theta^a} \right) = x_2^b - \frac{\theta^b}{(1 + \theta^b)} u(x_2^b) \tag{63}$$

The objective of the planner is proportional to

$$\begin{aligned} & u(x_2^b) - x_2^b + \tilde{d} \left(\rho - \frac{1}{\theta^a} \right) \\ &= u(x_2^b) - \frac{\theta^b}{(1 + \theta^b)} u(x_2^b) \end{aligned}$$

which is always increasing in x_2^b . Therefore deposits and transfers are essential if and only if $\rho\theta^a > 1$. This completes the proof.

6.5.4 $\theta^a < \bar{\theta}^a$ and $\theta^b < \bar{\theta}^b$ (sector b takes deposits)

In this case, the repayment constraint binds in both sector at the solution to (41). We now analyze the case where sector b takes deposits. We have the following result

Claim 15 *Deposits in sector b are Pareto essential if*

$$\theta^a < \bar{\theta}^a, \theta^b < \bar{\theta}^b \text{ and } \theta^a \rho < \left(1 - \frac{\theta^a}{\theta^b} \right) (1 + \rho).$$

When $\theta^b < \bar{\theta}^b$, the solution to (41) in sector b is \underline{x}_2^b , and the repayment constraint binds. Therefore, if agents in sector b receive a positive amount of deposits they will default. Deposits in sector b are incentive compatible only if agents in sector a also make a transfer τ to agents in sector b . We assume that transfers are made at the beginning of the period, stored and then consumed. Then, the repayment constraint in sector b with transfer τ and taking deposits d , evaluated at \underline{x}_2^b is

$$-\underline{x}_2^b - d + \theta^b \left[u(\underline{x}_2^b) - \underline{x}_2^b + \tau \right] \geq 0$$

By definition $\underline{x}_2^b + \theta^a \left[u(\underline{x}_2^b) - \underline{x}_2^b \right] = 0$ and the minimum transfer τ that keeps the constraint satisfied with deposits d is

$$\tau = d/\theta^b \tag{64}$$

The repayment constraint with transfers in sector a is

$$-(x_2^a - d) + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} - \tau \right] \geq 0 \tag{65}$$

Substituting (64) into (65), the repayment constraint in sector a gives

$$\theta^a u(x_2^a) - x_2^a \left(1 + \frac{\theta^a}{1 + \rho}\right) + d \left(1 - \frac{\theta^a \rho}{1 + \rho} - \frac{\theta^a}{\theta^b}\right) \geq 0$$

so, x_2^a will be increasing in d if

$$\theta^a \rho < \left(1 - \frac{\theta^a}{\theta^b}\right) (1 + \rho)$$

Notice that a agents' welfare is now proportional to

$$u(x_2^a) - \frac{x_2^a}{1 + \rho} - \frac{\rho}{1 + \rho} d - \tau \tag{66}$$

Substituting (64) into (66), as well as the value for d such that the repayment constraint is satisfied with equality, and maximizing we obtain that a agents' welfare is increasing d if and only if

$$u'(x_2^a) > 1 + \frac{1}{\theta^b}.$$

This requires that

$$u'(x_2^a) > 1 + \frac{1}{\theta^b} = u(x_2^b)/x_2^b > u'(x_2^b).$$

where the last inequality follows from concavity. Therefore we need $x_2^a < x_2^b$, or by (58)

$$\theta^a \rho \leq \left(1 - \frac{\theta^a}{\theta^b}\right) (1 + \rho)$$

Notice that in this economy the return on deposits for a agents is $-\frac{1}{\theta^b}$. Agents in sector a are willing therefore to forego the return $\rho > 0$ and actually pay b agents a positive amount in order to buy commitment and enjoy higher consumption.

Notice however that this result is possible only if deposits and transfers do not tighten the production constraint in sector a , making it binding, while relaxing the repayment constraint. Rewrite the production constraint and the repayment constraint in sector a respectively as

$$-\frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} - \tau + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} - \tau \right] \geq 0$$

$$-(x_2^a - d) + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} - \tau \right] \geq 0$$

Since $\tau = d/\theta^b$ the repayment constraint remains the relevant constraint whenever

$$x_2^a - d \geq \frac{x_2^a}{1 + \rho} + d \left(\frac{\rho}{1 + \rho} + \frac{1}{\theta^b} \right)$$

Since $\rho > 0$, this will be satisfied for any d small enough. This completes the proof.

6.5.5 $\theta^a < \bar{\theta}^a$ and $\theta^b \geq \bar{\theta}^b$

In this case, the repayment constraint at the solution to (41) in sector a binds but not in sector b . We have the following result

Claim 16 *Deposits in sector b are Pareto essential if*

$$\theta^a < \bar{\theta}^a, \theta^b \geq \bar{\theta}^b \text{ and } \theta^a \rho < 1 + \rho.$$

The proof follows. Agents in sector a forgo some return on storage whenever they deposit goods with agents in sector b . Therefore, given some allocation x_2^a and deposits d , they have to produce the amount y such that

$$x_2^a = (y - d)(1 + \rho) + d.$$

Then the production constraint in sector a become

$$-\frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} \right] \geq 0$$

and the repayment constraint is

$$-(x_2^a - d) + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} - d \frac{\rho}{1 + \rho} \right] \geq 0 \quad (67)$$

Notice that for small enough deposits, only the repayment constraint is the relevant constraint. Hence, to show that deposits are Pareto essential in sector b , we need to show that increasing d relax the repayment constraint in sector a . We can rewrite the repayment constraint in sector a with deposits d (67) as

$$-x_2^a + \theta^a \left[u(x_2^a) - \frac{x_2^a}{1 + \rho} \right] + d \left(1 - \frac{\theta^a \rho}{1 + \rho} \right) \geq 0 \quad (68)$$

Clearly, the left-hand side of (68) is increasing in d if and only if

$$\theta^a \rho < 1 + \rho.$$

Also, notice that the left-hand-side of (68) is decreasing in x_2^a . The derivative of the LHS with respect to x_2^a is

$$\theta^a u'(x_2^a) - \left(1 + \frac{\theta^a}{1 + \rho}\right)$$

from the definition of \underline{x}_2^a and concavity, we obtain

$$\frac{u(x_2^a)}{x_2^a} = \frac{1}{\theta^a} + \frac{1}{1 + \rho} > u'(\underline{x}_2^a)$$

Hence, for all $x > \underline{x}_2^a$, we have

$$\theta^a u'(x_2^a) - \left(1 + \frac{\theta^a}{1 + \rho}\right) < 0.$$

This shows that the left-hand side of (68) is decreasing in x_2^a . Hence, whenever $\theta^a \rho < 1 + \rho$, increasing deposits allows higher x_2^a .

Finally, we need to show that increasing deposits and losing the return on storage in sector a does not hurt agents in sector a too much. This requires that the total derivative of the welfare function is positive when evaluated at \underline{x}_2^a . Notice to this purpose that maximizing agents a welfare is equivalent to maximizing

$$u(x_2^a) - \frac{x_2^a}{1 + \rho} - \frac{\rho}{1 + \rho} d$$

Substituting the expression for deposits from (68) and maximizing with respect to x we see that welfare is increasing in x if

$$(1 + \rho)u'(\underline{x}^a) > \frac{1}{1 + \rho} - \theta^a \left(\frac{\rho}{1 + \rho}\right)^2$$

Since $u'(\underline{x}^a) > 1/(1 + \rho)$ we get that this condition is always satisfied. Therefore, the planner will want to increase d to relax the repayment constraint so as to increase x and achieve a higher welfare in sector a . This ends the proof.

7 References

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