

Accumulating success - a model of dynamic contests with a fixed endpoint*

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Abstract

We analyze a dynamic contest between two players who compete for a prize over a fixed number of stages. The player who is ahead at the end wins regardless of the history of the game. We find that if contestants are symmetric, their efforts will always be the same in the last stage of the contest. For all stages except the first and last equilibrium effort will typically not be symmetric. We find that under certain conditions the player who is in the lead has an additional incentive to invest in effort, resulting in a “favorite bias”. The intuition is that the player in the lead can, by increasing his effort, reduce future competition and thus future effort costs.

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Introduction

Contests have been analyzed extensively in the literature and much is known about the incentive effects of contests.¹ For an excellent recent overview over the literature consult Konrad (2007). Sisak (2009) surveys the literature on the optimal allocation of (multiple) prizes. Most of this literature focuses on static one-shot contests or simple two-stage set-ups. This is in contrast to reality where in many contests we observe dynamic features. This paper attempts to fill in the void. We analyze a dynamic contest between two players who compete for a prize over a fixed number of stages. The player who is ahead at the end wins regardless of the history of the game. Hence, we model the contest as an All-pay auction where players spend efforts not only once but possibly over a span of time or at certain points in time. This structure can be found, for example, in promotion tournaments and elections, but also in sporting events like football or icehockey.

Our paper is most closely related to Yildirim (2005) and Lazear and Rosen (1981). Yildirim (2005) analyzes a two stage contest with constant marginal costs and a Tullock contest success function. We extend his paper to an arbitrary number of stages, a fully discriminating contest success function and assume that marginal costs are increasing with effort (convex). Lazear and Rosen (1981) model a contest in which players spend effort in order to win a single prize. The contest technology is noisy in the respect that there is an additive noise term. The player with the highest effective effort, i.e effort plus noise, wins the contest. In contrast to their paper we allow for repeated interaction. Our paper is also related to the work of Harris and Vickers (1985), Harris and Vickers (1987) on dynamic races and tug-o-wars in discrete time and the paper of Moscarini and Smith (2008) analyzing a dynamic tug-o-war in continuous time. Other papers analyzing dynamic contests in continuous time are Eggert et al. (2007), Amegashie and Runkel (2008) and Wirl (1994). Similar papers from fields other than contest or auction theory are for example Reinganum (1981), who analyzes dynamic R&D races, and the paper by Athey and Schmutzler (2001) on market dominance.

¹Early work has been done by for example Tullock (1980) with an application to lobbying. Other examples are the analysis of sporting events (e.g. Szymanski (2003)), R&D races (e.g. Baye and Hoppe (2003)), the provision of public goods (Kolmar and Sisak (2008)), property rights (e.g. Grossman and Kim (1995), Garfinkel and Skaperdas (2007), Kolmar (2007)), or litigation (e.g. Gürtler and Kräkel (2008)).

We assume that contestants' effort is disturbed by independent, additive shocks in every stage. These shocks can be interpreted as luck or generally as events which are not under the control of the contestants. In most cases we assume those shocks to be normally distributed with mean zero, to avoid to have to deal with convolutions when solving for closed form solutions.

We first solve the game for the open loop equilibrium, where players determine their respective stage efforts before the game starts. This has some advantages, but also some shortcomings, since players cannot dynamically adjust their strategies to the history of play. Therefore we also solve for the closed loop or feedback equilibrium of the game.² We model the closed loop game as a Markov game. This implies that current play only depends on the current state but not on the exact history of the game. In our case this state variable is defined as the lead of player one (if it is negative, player one is behind). We find that if contestants are symmetric, their efforts will always be the same in the last stage of the contest. They will be increasing in the value of the prize, decreasing in the absolute value of the lead of player one and non-monotonous in the noise parameter. If the contest is very close then more noise decreases effort as additional effort is less effective. If one player is way ahead and there is only little noise increasing noise can actually increase effort. Increasing noise gives the player who is far behind a more realistic chance to catch up. Competition is increased and both players will invest more. At some point though the effect described above kicks in and more noise will only decrease the incentive to invest in effort as it waters down incentives. The last stage in isolation can be seen as a one-shot contest with one player having a head start. Hence even though there is asymmetry because one player is ahead they still invest the same amount of effort. This finding is similar to the equilibrium in a Tullock contest when contestants are heterogeneous with respect to their efforts effectiveness (e.g. Konrad (2007)).

For all stages except the last stage equilibrium efforts change qualitatively. There are four effects which influence the choice of efforts, two direct effects, well-known from the one-shot contest, and two indirect effects. The direct effects are the direct costs and direct benefits of effort. They will be symmetric for both players. The additional indirect effects are on the

²See, for example, ? for a discussion of those equilibrium concepts.

one hand the effect of effort now on later stage efforts and through this on the probability of success in the last stage. This effect is symmetric for both players, though it is not possible to quantify the effect exactly. On the other hand we have the indirect effect of effort now on later effort and through this on later costs. This effect is indeterminate and asymmetric. Hence this effect is responsible that effort will typically not be the same for both players. We find that under certain conditions this effect will favor the player who is in the lead, resulting in a “favorite bias”. The intuition is that the player in the lead can, by increasing his effort, reduce future competition and thus future effort costs. The player who is behind in contrast will by investing more increase competition in the future and hence increase future expected costs. This finding is similar to the increasing dominance result in Athey and Schmutzler (2001) and is a testable prediction of our model.

General Setup

We analyze a contest that consists of T stages. In every stage players can invest effort in order to increase their chances to win the prize R , which the value equally. In every stage players’ efforts are disturbed by an additive shock, which is iid across stages and distributed according to the density $g(\cdot)$. We assume $g(\cdot)$ symmetric around zero, which implies that luck is neutral or “fair”. As mentioned above, in most cases we assume those shocks to be normally distributed with mean zero and variance $\sigma^2/2$, to avoid to have to deal with convolutions when solving for closed form solutions.³

We assume that players’ costs $C(x)$ are increasing in x , convex, differentiable, identical for both players and identical across all stages t , so that $C_{1t}(x) = C_{2t}(x) = C_t(x) = C(x)$. We abstain from assuming a discount rate at this moment, so that efforts across stages are perfect substitutes. Both players are risk neutral.

³For example, when in every stage noise is normal with mean zero and variance $\sigma^2/2$, the cumulative noise is normal with mean zero and variance $T\sigma^2$. Without the normality assumption it would be a real mess to calculate the cumulative noise.

Open loop equilibrium

In an open loop equilibrium all players specify their respective strategies before the game begins. Players do not update or adjust their strategies during the game. The expected utility of player 1 is

$$EU_1 = G_{cum} \left(\sum_{t=1}^T (x_{1t} - x_{2t}) \right) R - \sum_{t=1}^T C(x_{1t}) \quad (1)$$

while player 2's expected utility is

$$EU_2 = \left[1 - G_{cum} \left(\sum_{t=1}^T (x_{1t} - x_{2t}) \right) \right] R - \sum_{t=1}^T C(x_{2t}). \quad (2)$$

$G_{cum}(\cdot)$ is the distribution of the sum of T shocks distributed according to $g(\cdot)$. The density of the cumulative shocks is $g_{cum}(\cdot)$. Then, player i 's first order conditions with respect to effort in period t are

$$\frac{\partial EU_i}{\partial x_{it}} = g_{cum} \left(\sum_{t=1}^T (x_{1t}^* - x_{2t}^*) \right) R - C'(x_{it}^*) \geq 0.$$

Now player i 's FOCs are identical for all stages, such that effort is identical through all stages and given by $x_{it}^* = x_i^* \forall t$. Moreover, because both players' FOCs are also identical by the same reasoning $x_1^* = x_2^* = x^*$. Therefore, the FOCs simplify to

$$g_{cum}(0) R - C'(x^*) \geq 0.$$

Assuming an interior solution exists the equilibrium effort is characterized by

$$x^* = C'^{-1}(g_{cum}(0) R).$$

This effort is increasing in R and $g_{cum}(0)$ and decreases in the convexity of the cost function. Equilibrium expected utility is given by

$$EU_i^* = \frac{R}{2} - TC(C'^{-1}(g_{cum}(0)R)).$$

If for example $C(x) = \frac{x^2}{2}$ and $G_{cum}(\cdot)$ is normal with variance $T\sigma^2$ and mean zero, we find

$$x^* = \frac{R}{\sigma\sqrt{2T\pi}}.$$

Closed loop equilibrium

Despite the simplicity and the intuitive appeal of the open loop solution, we think this is not the way people actually behave in many situations. In open loop equilibria players do not adjust their strategies after observing the realization of the noise in earlier periods. This might be reasonable in some situations, in particular when the realization of the noise is not observable. Nevertheless, in this section we analyze the game when players adjust their strategies after they realized the noise in the past.

To analyze the role of dynamics and interaction we start with the simplest possible model. There are two players, named 1 and 2, who compete over a given number of periods T for a prize which is worth R to both of them. In each period they can expend effort x_{it} which translates into progress in the contest, though only noisily. We assume that each contestant's effort is hit in each period by a random, independent and normally distributed shock ϵ_{it} . This represents all the factors which are not under the control of the two players like for example luck or weather. In each period the state of the game is characterized completely by one state variable which captures the history of play and is named the lead of player one $d_t = \sum_{i=1}^t (x_{1i} + \epsilon_{1i} - x_{2i} - \epsilon_{2i})$. We abbreviate notation by defining $\Delta x_t = x_{1t} - x_{2t}$ as well as $\epsilon_{1t} - \epsilon_{2t} = \Delta \epsilon_t$. So efforts in all periods are perfect substitutes. This implies that in the end it does not matter at all when a player expended his effort or how often he has been in the lead. There will be infinitely many histories all leading to the same outcome. In each period players face quadratic costs of effort. This has two reasons: firstly, a convex cost

function guarantees equilibria where both players spend positive efforts. Also, the quadratic specification makes the model analytically more tractable and is widely used in the literature (for example, Moscarini and Smith (2008) and Amegashie and Runkel (2008)). This form gives the players an incentive to spread efforts evenly over all rounds in the absence of any further effects as we showed in the open loop equilibrium. Hence it will be interesting to see when and how the players deviate from this allocation.

The expected utility at the beginning of the game is equal to

$$EU_{i1} = Pr \left[\sum_{t=1}^T (x_{it} + \epsilon_{it}) > \sum_{t=1}^T (x_{jt} + \epsilon_{jt}) \right] R - \sum_{t=1}^T \frac{x_{it}^2}{2}; \quad i, j = 1, 2; \quad i \neq j.$$

We can characterize the probability more exactly. Let us assume that each ϵ_{it} is distributed normally with mean zero and variance $\sigma^2/2$. Then $\Delta\epsilon$ is distributed normally with mean zero and variance σ^2 . Furthermore $\sum_{t=1}^T \Delta\epsilon_t$, the difference of accumulated shocks, is distributed normally with mean zero and variance $\sigma^2 T$ because we have a sum of identically and independently distributed normal random variables. Let $G_t(\cdot)$ stand for the cumulative distribution function of a normal random variable with mean zero and variance $(T + 1 - t)\sigma^2$. Hence $G_T(\cdot)$, the uncertainty in the last period, has a variance of only σ^2 while $G_1(\cdot)$, uncertainty in the first period, has a variance of $T\sigma^2$. We can rewrite expected utility as

$$\begin{aligned} EU_{11} &= G_1 \left[\sum_{t=1}^T \Delta x_t \right] R - \sum_{t=1}^T \frac{x_{1t}^2}{2} \\ EU_{21} &= \left(1 - G_1 \left[\sum_{t=1}^T \Delta x_t \right] \right) R - \sum_{t=1}^T \frac{x_{2t}^2}{2}. \end{aligned}$$

Note that the expected utility function is not necessarily concave. Hence in order for the second order conditions for a utility maximum to hold we will need that either there is sufficient noise and hence σ^2 is sufficiently large or the value of the prize R to be sufficiently small. Hence for a dynamic contest with little noise and a high prize there will be corner solutions and mixed strategies. Although those games are quite interesting to analyze, we put our focus on contests with interior solutions, where both players spend positive efforts in all stages. In the next section we start by solving the game backwards and we look more

closely at the last stage of the contest, stage T .

The last stage

Expected utility in the last period is equal to

$$\begin{aligned} EU_{1T} &= G_T [\Delta x_T + d_T] R - \frac{x_{1T}^2}{2} \\ EU_{2T} &= (1 - G_T [\Delta x_T + d_T]) R - \frac{x_{2T}^2}{2}. \end{aligned}$$

where d_T is our state variable and is equal to the lead in period T of player one. This stage is similar to Lazear and Rosen (1981) with individual handicaps. We can see that an increase in d_T ceteris paribus increases player one's probability of success given arbitrary effort investments by both players. The effort costs of all previous periods are sunk. The first order conditions in period T are given by

$$\begin{aligned} \frac{\partial EU_{1T}}{\partial x_{1T}} &= g_T [\Delta x_T + d_T] R - x_{1T} \stackrel{!}{=} 0 \\ \frac{\partial EU_{2T}}{\partial x_{2T}} &= g_T [\Delta x_T + d_T] R - x_{2T} \stackrel{!}{=} 0. \end{aligned}$$

We can state our first result:

Proposition 1. *Last stage effort will always be symmetric. Given an interior solution exists effort is equal to*

$$x_{1T}(d_T) = x_{2T}(d_T) = x_T(d_T) = g_T[d_T]R. \quad (3)$$

Given that $g_T(\cdot)$ is the normal density we have

$$x_T(d_T) = \frac{e^{-\frac{d_T^2}{2\sigma^2}} R}{\sigma\sqrt{2\pi}}.$$

Proof. Follows directly by the first order conditions inserting the density of the normal distribution and noting that $\Delta x_T(d_T) = 0$ in a symmetric equilibrium. \square

Last stage effort is linearly increasing in the value of the prize R , decreasing in the absolute

value of lead d_T (maximum effort is reached when $d_T = 0$) and non-monotonous in noise σ^2 . Effort is increasing in noise for $\sigma^2 < d_T^2$ and decreasing after. To make the intuition clear let us assume first that the contest is very close in the last stage and hence d_T close to zero. In this case effort is always decreasing in noise as $\sigma^2 > d_T^2$. This is the case because noise distorts the marginal benefit of investment leading to a smaller increase in probability of success. In contrast now assume d_T is very large and hence for example player two is hopelessly trailing behind. Also assume that noise is low and hence $\sigma^2 < d_T^2$. In this case player two is discouraged to invest anything much because he has no chance to catch up and hence player one can win without needing to fight much. An increase in noise here will make it easier for player two to catch up and hence will motivate him to invest more. This in turn leads to player one investing more to defend his lead. Once noise becomes too large again the effect discussed first kicks in again and noise distorts the incentive to invest more effort.

Note that the game discussed so far - the last stage of a dynamic contest - can also be interpreted as a one-shot game with head start. Hence even though we have an asymmetry created by giving one player a head start both players will invest symmetrically in the one-shot model. In the next section we will see how this result changes once we include simple dynamics.

Stages $t < T$

When deciding on effort in the last stage T players face a simple trade off. On the one hand more effort will ceteris paribus increase the probability of success. On the other hand more effort increases the costs. Those two effects are symmetric for both players and hence they choose identical efforts. The decision in all other stages is more complicated. We have two additional effects. An increase in effort now has an effect on the lead in later periods and through this on effort in later periods. This will on the one hand influence the probability of success and on the other hand future costs of effort. We now show that these additional effects make the contest asymmetric in all stages $t < T$.

Let us illustrate these effects first by looking at stage $T - 1$. The first order conditions

are⁴

$$\begin{aligned}\frac{\partial EU_{1T-1}}{\partial x_{1T-1}} &= E_{T-1} \left[g_T(\Delta x_T + \Delta x_{T-1} + \Delta \epsilon_{T-1} + d_{T-1}) \left(1 + \frac{\partial \Delta x_T}{\partial d_T} \frac{\partial d_T}{\partial x_{1T-1}} \right) \right] R \\ &\quad - x_{1T-1} - E_{T-1} \left[x_{1T} \frac{\partial x_{1T}}{\partial d_T} \frac{\partial d_T}{\partial x_{1T-1}} \right] \stackrel{!}{=} 0 \\ \frac{\partial EU_{2T-1}}{\partial x_{2T-1}} &= -E_{T-1} \left[g_T(\Delta x_T + \Delta x_{T-1} + \Delta \epsilon_{T-1} + d_{T-1}) \left(-1 + \frac{\partial \Delta x_T}{\partial d_T} \frac{\partial d_T}{\partial x_{2T-1}} \right) \right] R \\ &\quad - x_{2T-1} - E_{T-1} \left[x_{2T} \frac{\partial x_{2T}}{\partial d_T} \frac{\partial d_T}{\partial x_{2T-1}} \right] \stackrel{!}{=} 0,\end{aligned}$$

where we already know that $\Delta x_T = 0$, $\frac{\partial \Delta x_T}{\partial d_T} = 0$ by symmetry in the last period (see proposition 1) as well as $\frac{\partial d_T}{\partial x_{1T-1}} = 1$ and $\frac{\partial d_T}{\partial x_{2T-1}} = -1$. Note that the marginal benefit will be the same for both players, both the direct and the indirect effect.

$$E_{T-1} \left[\underbrace{g_T(\Delta x_{T-1} + \Delta \epsilon_{T-1} + d_{T-1})}_{\text{direct effect}} + \underbrace{g_T(\Delta x_{T-1} + \Delta \epsilon_{T-1} + d_{T-1}) \overbrace{\frac{\partial \Delta x_T}{\partial d_T}}^{=0}}_{\text{indirect effect}} \right] R$$

Again, the direct effect is how x_{iT-1} directly influences the probability of success in stage T . The indirect effect is how x_{iT-1} influences the probability of success in stage T indirectly over its influence on effort in stage T . Furthermore we know that in stage $T-1$ this indirect effect on effort will be zero due to symmetry in the last stage. Hence our first-order conditions simplify to

$$\begin{aligned}\frac{\partial EU_{1T-1}}{\partial x_{1T-1}} &= g_{T-1}(\Delta x_{T-1} + d_{T-1})R - \underbrace{x_{1T-1}}_{\text{direct effect}} - \underbrace{E_{T-1} \left[x_{1T} \frac{\partial x_{1T}}{\partial d_T} \right]}_{\text{indirect effect}} \stackrel{!}{=} 0 \quad (4) \\ \frac{\partial EU_{2T-1}}{\partial x_{2T-1}} &= g_{T-1}(\Delta x_{T-1} + d_{T-1})R - x_{2T-1} - E_{T-1} \left[x_{2T} \frac{\partial x_{2T}}{\partial d_T} (-1) \right] \stackrel{!}{=} 0,\end{aligned}$$

where we have already highlighted the relevant cost effects. The direct cost effect is always symmetric as we assume that both players have the same cost function. The indirect effect is more interesting. It represents the effect of effort x_{iT-1} on costs in T . Note that effort in

⁴For notational brevity we omit that effort depends on the state, e.g. $\Delta x_T = \Delta x_T(d_T)$ in the following discussion.

T is a random variable from the viewpoint of period $T - 1$ as $\Delta\epsilon_{T-1}$ enters the expression through lead d_T . The indirect effect is symmetric for

$$\begin{aligned} E_{T-1} \left[x_{1T} \frac{\partial x_{1T}}{\partial d_T} \right] &= -E_{T-1} \left[x_{2T} \frac{\partial x_{2T}}{\partial d_T} \right] \\ \Leftrightarrow E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right] &= -E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right] \\ \Leftrightarrow 2E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right] &= 0, \end{aligned}$$

where we make use of the symmetry in stage T . Intuitively the indirect effect on future costs has to be zero in expectation for both players. The following lemma determines the indirect cost effect depending on d_{T-1} :

Lemma 1. *The indirect effect on costs in stage T of stage $T - 1$ effort is symmetric if and only if $d_{T-1} = 0$. In this case it is equal to zero for both players.*

Proof. See appendix. □

For $d_{T-1} = 0$ it is equally likely that player one will be behind as it is that player two will be behind in the next period given that they invest symmetrically. The game is totally symmetric. Increasing the lead today will have indirect benefits as often as it will have indirect costs and it will cancel in expectation for both players. If, on the other hand, one player is ahead it is more likely that he also stays ahead in the next period, given equal investment in $T - 1$. Hence he has a greater incentive to invest in effort to reduce future effort costs as effort in T is hump-shaped around zero. The following proposition summarizes this intuition.

Proposition 2. *Effort in stage $T - 1$ will be symmetric for $d_{T-1} = 0$. If $d_{T-1} > 0$ ($d_{T-1} < 0$) player one (two) will invest more than player two (one).*

Proof. Follows directly from Lemma 1 and the FOCs in (4) as well as the SOC's which we assume to hold. □

The intuition behind Proposition 2 is that if player one is ahead an increase in effort is likely to lead to less competition in the future as he can increase his lead. On the other hand if

player one is behind an increase of effort will likely increase competition and hence future effort costs because he will catch up with his opponent in expectation.

We cannot present the equilibrium effort in closed form for all cases but we can state some interesting properties and can solve a special case. When effort is symmetric, i.e. when $d_{T-1} = 0$, effort is given by

$$x_{T-1}(0) = g_{T-1}(0)R = \frac{R}{2\sqrt{2\pi}\sigma},$$

which is exactly half of the effort in stage T given the same value of the state variable ($x_{T-1}(0) = \frac{x_T(0)}{2}$). This is due to the increase in noise which discourages investment with symmetric types *ceteris paribus*. We also know that efforts of player one and player two must be mirror symmetric, meaning that $x_{1T-1}(d_{T-1}) = x_{2T-1}(-d_{T-1})$. This is the case because d_{T-1} has exactly the same value for player 1 as $-d_{T-1}$ for player 2, something that should be intuitively clear. Furthermore we can show that the derivative of the first order conditions with respect to d_{T-1} at the symmetric equilibrium is positive for player one and negative for player two.⁵

$$\begin{aligned} \frac{\partial^2 EU_{1T-1}}{\partial x_{1T-1} \partial d_{T-1}} &= g'_{T-1}(\Delta x_{T-1} + d_{T-1})R - \frac{\partial E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right]}{\partial d_{T-1}} \\ \frac{\partial^2 EU_{2T-1}}{\partial x_{1T-1} \partial d_{T-1}} &= g'_{T-1}(\Delta x_{T-1} + d_{T-1})R + \frac{\partial E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right]}{\partial d_{T-1}} \end{aligned}$$

At the symmetric equilibrium at $d_{T-1} = 0$ the first term is equal to zero, because $g(\cdot)$ is symmetric around zero. Therefore only the indirect cost effect determines the sign and size of this derivative in this case. This is easily calculated and is positive for player one in zero and negative for player two. This means that in contrast to stage T maximum effort is not expended when the state is zero.

There exists one further value of $d_{T-1} > 0$ for player 1 such that he optimally chooses the same effort as in the symmetric equilibrium at $d_{T-1} = 0$. Player two who trails behind

⁵As we assume the second order condition to hold throughout $\frac{\partial^2 EU_{iT-1}}{\partial x_{iT-1}^2} < 0$. In other words, at the optimum, the first order condition is decreasing in own effort.

in this case expends less than player one. This is of course also true vice versa. Hence player 1's (player 2's) effort in $T - 1$ is hump-shaped and gets maximal when $d_{T-1} > 0$ ($d_{T-1} < 0$). Thus we find a “favorite bias”, since the favorite, i.e. the player in the lead, has a stronger incentive to invest due to the indirect cost effect and hence in expectation will increase his lead. This finding is similar to the increasing market dominance result in Athey and Schmutzler (2001). They analyze a market with oligopolistic competition and ongoing investment. In this framework they identify circumstances under which a firm with lower costs or higher product quality has an incentive to extend its initial lead through investments.

Having characterized the last two stages of the game we now go on and analyze the general problem across stages. First we show that the expected marginal benefit of effort will always be equal for both players, as it was in stages $T - 1$ and T .

Lemma 2. *The expected marginal benefit of effort is symmetric for the two players in all periods $t = 1, \dots, T$.*

Proof. This follows from the fact that our game is a Markov game and that our state variable is just a sum of unweighed differences of efforts and noise. In our notation we suppress the dependence of effort on previous efforts in order to keep the exposition clearer. Marginal benefit for both players is given by

$$\begin{aligned} \text{MB}_1^t &= E_{t,T-1} \left[g_T \left(\sum_{i=t}^T (\Delta x_i + \Delta \epsilon_i) + d_t \right) \Lambda_{1t} \right] R \\ \text{MB}_2^t &= E_{t,T-1} \left[-g_T \left(\sum_{i=t}^T (\Delta x_i + \Delta \epsilon_i) + d_t \right) \Lambda_{2t} \right] R \end{aligned}$$

where we define $E_{t,k}[\dots]$ as the expectation operator over noise ($\Delta \epsilon$) in periods t to k and Λ_{1t} and Λ_{2t} are the respective inner derivatives. They are equal to

$$\begin{aligned} \Lambda_{1t} &= 1 + \sum_{i=1}^{T-t} \frac{\partial \Delta x_{t+i}}{\partial d_{t+i}} \left(\frac{\partial d_{t+i}}{\partial x_{1t}} + \sum_{j=1}^{i-1} \frac{\partial \Delta x_{t+i-j}}{\partial d_{t+i-j}} \frac{\partial d_{t+i-j}}{\partial x_{1t}} \prod_{k=1}^{j-1} \left(1 + \frac{\partial \Delta x_{t+i-j+k}}{\partial d_{t+i-j+k}} \right) \right) \\ \Lambda_{2t} &= -1 + \sum_{i=1}^{T-t} \frac{\partial \Delta x_{t+i}}{\partial d_{t+i}} \left(\frac{\partial d_{t+i}}{\partial x_{2t}} + \sum_{j=1}^{i-1} \frac{\partial \Delta x_{t+i-j}}{\partial d_{t+i-j}} \frac{\partial d_{t+i-j}}{\partial x_{2t}} \prod_{k=1}^{j-1} \left(1 + \frac{\partial \Delta x_{t+i-j+k}}{\partial d_{t+i-j+k}} \right) \right). \end{aligned}$$

Note that $\Lambda_1 = -\Lambda_2$ because $\frac{\partial d_t}{\partial x_{1t}} = -\frac{\partial d_t}{\partial x_{2t}}$. \square

Here we see that since both players' effort influences future effort only through the state variable and only the difference in efforts is relevant for the probability of success, their marginal benefit from investing is identical.

Let us take a look at the players' expected marginal costs of effort. They can be represented as

$$\begin{aligned} \text{MC}_1^t &= x_{1t} + \sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{1t+j} \frac{\partial x_{1t+j}}{\partial d_{t+j}} \left(\frac{\partial d_{t+j}}{\partial x_{1t}} + \sum_{k=1}^{j-1} \frac{\partial d_{t+j-k}}{\partial x_{1t}} \frac{\partial \Delta x_{t+j-k}}{\partial d_{t+j-k}} \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}}{\partial d_{t+j-k+i}} \right) \right) \right] \\ &= x_{1t} + \sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{1t+j} \frac{\partial x_{1t+j}}{\partial d_{t+j}} \left(1 + \sum_{k=1}^{j-1} \frac{\partial \Delta x_{t+j-k}}{\partial d_{t+j-k}} \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}}{\partial d_{t+j-k+i}} \right) \right) \right] \end{aligned}$$

and

$$\begin{aligned} \text{MC}_2^t &= x_{2t} + \sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{2t+j} \frac{\partial x_{2t+j}}{\partial d_{t+j}} \left(\frac{\partial d_{t+j}}{\partial x_{2t}} + \sum_{k=1}^{j-1} \frac{\partial d_{t+j-k}}{\partial x_{2t}} \frac{\partial \Delta x_{t+j-k}}{\partial d_{t+j-k}} \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}}{\partial d_{t+j-k+i}} \right) \right) \right] \\ &= x_{2t} + \sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{2t+j} \frac{\partial x_{2t+j}}{\partial d_{t+j}} \left(-1 - \sum_{k=1}^{j-1} \frac{\partial \Delta x_{t+j-k}}{\partial d_{t+j-k}} \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}}{\partial d_{t+j-k+i}} \right) \right) \right] \\ &= x_{2t} - \sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{2t+j} \frac{\partial x_{2t+j}}{\partial d_{t+j}} \left(1 + \sum_{k=1}^{j-1} \frac{\partial \Delta x_{t+j-k}}{\partial d_{t+j-k}} \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}}{\partial d_{t+j-k+i}} \right) \right) \right], \end{aligned}$$

where we define $E_{t,t+j}[\dots]$ as the expectation operator over noise ($\Delta\epsilon$) in periods t to $t+j$. We see that the indirect cost effects typically work in opposite directions creating a bias. So only if the sum of indirect expected marginal costs is equal for both players, we have a symmetric equilibrium. For $t = T - 1$ we have shown that this arises only when no player starts the stage with a lead and hence $d_{T-1} = 0$. Also the indirect costs will be zero in this case.

Because the asymmetry between players is summarized by the state variable efforts are always mirror symmetric in the sense that $x_{1t}(d_t) = x_{2t}(-d_t)$. This implies that $\frac{\partial x_{1t}}{\partial d_t}(d_t) = -\frac{\partial x_{2t}}{\partial d_t}(-d_t)$ are point symmetric and hence $x_{1t}(d_t) \frac{\partial x_{1t}}{\partial d_t}(d_t) = -x_{2t}(-d_t) \frac{\partial x_{2t}}{\partial d_t}(-d_t)$ are also point symmetric around zero. What about $\frac{\partial \Delta x_t}{\partial d_t}(d_t)$? By mirror symmetry $\Delta x_t(d_t) = -\Delta x_t(-d_t)$ and $\frac{\partial \Delta x_t}{\partial d_t}(d_t) = \frac{\partial \Delta x_t}{\partial d_t}(-d_t)$. Using these facts we can rewrite the indirect marginal costs for player two in the following way

$$\sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{1t+j}(-d_{t+j}) \frac{\partial x_{1t+j}}{\partial d_{t+j}}(-d_{t+j}) \left(1 + \sum_{k=1}^{j-1} \frac{\partial \Delta x_{t+j-k}}{\partial d_{t+j-k}}(-d_{t+j-k}) \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}}{\partial d_{t+j-k+i}}(-d_{t+j-k+i}) \right) \right) \right].$$

Here we can clearly see that it will be equal to the indirect marginal costs of player one

$$\sum_{j=1}^{T-t} E_{t,t+j-1} \left[x_{1t+j}(d_{t+j}) \frac{\partial x_{1t+j}(d_{t+j})}{\partial d_{t+j}} \left(1 + \sum_{k=1}^{j-1} \frac{\partial \Delta x_{t+j-k}(d_{t+j-k})}{\partial d_{t+j-k}} \prod_{i=1}^{k-1} \left(1 + \frac{\partial \Delta x_{t+j-k+i}(d_{t+j-k+i})}{\partial d_{t+j-k+i}} \right) \right) \right]$$

if noise is such that it is symmetric around $d_k = \sum_{i=1}^{k-1} (\Delta x_i(d_i) + \Delta \epsilon_i) = 0$. Hence we only need to show that the random variable d_k is distributed symmetrically around zero. Let us examine this expression a little closer. We can distinguish three building blocks. First since we can be at an arbitrary point in time which is summarized by state variable d_t which is exogenous. Second we have pure noise which is symmetric and has mean zero $\sum_{i=t}^{k-1} \Delta \epsilon_i$. Third we have to consider our anticipated future actions which are a random variable when deciding on effort in t . We know that $\Delta x_j(d_j)$ is point symmetric around zero. Hence if the expectation of d_j is zero, the expectation of $\Delta x_j(d_j)$ will also be zero and the distribution will be symmetric. Hence as we should expect if $d_t = 0$ then the indirect effects are of the same magnitude and sign and the players choose the same action.

Let us now assume that player one is slightly in the lead and hence $d_t > 0$. Then *ceteris paribus* the lead one period later will in expectation be positive as well and the expectation of $\Delta x_{t+j}(d_{tj})$ will also become positive.

... to be continued...

Extensions

General cost function $c(x)$. General distribution function.

Correlated noise, IO Lit, Sub- and supermodular games...

Conclusion

In this paper we made a first attempt at opening the black-box of a one-shot contest model and look at dynamics and interaction. We find that effort in the last stage will always be symmetric even though typically one player will have an advantage over the other. This changes dramatically in all stages before where very generally the players invest different

amounts of effort. We identify a favorite bias. Typically the player who is in the lead has an increased incentive to invest in effort due to a favorable indirect cost effect.

Appendix

A Proof of Lemma 1

Let us plug in the optimal effort and simplify to determine the root of the indirect cost effect:

$$\begin{aligned}
2E_{T-1}\left[x_T \frac{\partial x_T}{\partial d_T}\right] &= 0 \\
\Leftrightarrow E_{T-1}\left[-\frac{d_T R^2}{2\pi(\sigma^2)^2} e^{-\frac{d_T^2}{\sigma^2}}\right] &= 0 \\
\Leftrightarrow E_{T-1}\left[-d_T e^{-\frac{d_T^2}{\sigma^2}}\right] &= 0 \\
\Leftrightarrow -\int_{-\infty}^{\infty} \frac{d_{T-1} + \Delta x_{T-1} + z}{\sqrt{2\pi}\sigma} e^{-\frac{z^2 + 2(d_{T-1} + \Delta x_{T-1} + z)^2}{2\sigma^2}} dz &= 0 \\
\Leftrightarrow -\int_{-\infty}^{\infty} (d_{T-1} + \Delta x_{T-1} + z) e^{-\frac{z^2 + 2(d_{T-1} + \Delta x_{T-1} + z)^2}{2\sigma^2}} dz &= 0
\end{aligned}$$

Note that $-d_T e^{-\frac{d_T^2}{\sigma^2}}$ is point symmetric around zero. Intuitively with symmetric noise centered around d_{T-1} the expectation will only be zero for $d_{T-1} = 0$. We prove this by continuing the calculations:

$$-\left[\frac{\left((d_{T-1} + \Delta x_{T-1}) e^{\frac{(3z + 2(d_{T-1} + \Delta x_{T-1}))^2}{6\sigma^2}} \sqrt{6\pi} \operatorname{Erf}\left(\frac{3z + 2(d_{T-1} + \Delta x_{T-1})}{\sqrt{6}\sigma}\right) - 6\sigma \right)}{e^{\frac{3z^2 + 4(d_{T-1} + \Delta x_{T-1})z}{2\sigma^2}}} \right]_{-\infty}^{\infty} = 0$$

As we are looking for a symmetric equilibrium in $T - 1$ we assume Δx_{T-1} will be zero and verify below that this is indeed optimal for a certain d_T .

$$\begin{aligned}
-\left[\frac{\left(d_{T-1} e^{\frac{(3z + 2d_{T-1})^2}{6\sigma^2}} \sqrt{6\pi} \operatorname{Erf}\left(\frac{3z + 2d_{T-1}}{\sqrt{6}\sigma}\right) - 6\sigma \right)}{e^{\frac{3z^2 + 4d_{T-1}z}{2\sigma^2}}} \right]_{-\infty}^{\infty} &= 0 \\
\Leftrightarrow -\left[\left(d_{T-1} e^{\frac{2d_{T-1}^2}{3\sigma^2}} \sqrt{6\pi} \operatorname{Erf}\left(\frac{3z + 2d_{T-1}}{\sqrt{6}\sigma}\right) \right) \right]_{-\infty}^{\infty} &= 0 \\
\Leftrightarrow -2d_{T-1} e^{\frac{2d_{T-1}^2}{3\sigma^2}} \sqrt{6\pi} &= 0
\end{aligned}$$

We see that the last equation can only be fulfilled for $d_{T-1} = 0$. □

B Comparative statics in $T - 1$

In order to do some comparative statics we totally differentiate the first order conditions which we first simplify as much as possible:

$$\begin{aligned}\frac{\partial EU_{1T-1}}{\partial x_{1T-1}} &= g_{T-1}(\Delta x_{T-1} + d_{T-1})R - x_{1T-1} - E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right] \stackrel{!}{=} 0 \\ \frac{\partial EU_{2T-1}}{\partial x_{2T-1}} &= g_{T-1}(\Delta x_{T-1} + d_{T-1})R - x_{2T-1} + E_{T-1} \left[x_T \frac{\partial x_T}{\partial d_T} \right] \stackrel{!}{=} 0.\end{aligned}$$

Here is the total derivative:

$$\begin{aligned}0 &= g'_{T-1}(\Delta x_{T-1} + d_{T-1})R(dx_{1T-1} - dx_{2T-1} + dd_{T-1}) - dx_{1T-1} \\ &\quad - E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right] (dx_{1T-1} - dx_{2T-1} + dd_{T-1}) \\ 0 &= g'_{T-1}(\Delta x_{T-1} + d_{T-1})R(dx_{1T-1} - dx_{2T-1} + dd_{T-1}) - dx_{2T-1} \\ &\quad + E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right] (dx_{1T-1} - dx_{2T-1} + dd_{T-1}).\end{aligned}$$

Rearranging endogenous and exogenous variables and denoting $g'_{T-1}(\Delta x_{T-1} + d_{T-1})$ by $g'_{T-1}(\cdot)$, we get

$$\begin{aligned}& \left(g'_{T-1}(\cdot)R - E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right] \right) (dx_{1T-1} - dx_{2T-1}) - dx_{1T-1} \\ &= \left(E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right] - g'_{T-1}(\cdot)R \right) dd_{T-1} \\ & \left(g'_{T-1}(\cdot)R + E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right] \right) (dx_{1T-1} - dx_{2T-1}) - dx_{2T-1} \\ &= - \left(E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right] + g'_{T-1}(\cdot)R \right) dd_{T-1}.\end{aligned}$$

Now we can calculate the partial effects using Cramer's rule

$$\frac{\partial x_{1T-1}}{\partial d_{T-1}} = \frac{g'_{T-1}(\cdot)R - E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}{1 + 2E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}$$

$$\frac{\partial x_{2T-1}}{\partial d_{T-1}} = \frac{g'_{T-1}(\cdot)R + E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}{1 + 2E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}.$$

At $d_{T-1} = 0$ this simplifies to

$$\frac{\partial x_{1T-1}}{\partial d_{T-1}} = \frac{-E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}{1 + 2E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}$$

$$\frac{\partial x_{2T-1}}{\partial d_{T-1}} = \frac{E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}{1 + 2E_{T-1} \left[\left(\frac{\partial x_T}{\partial d_T} \right)^2 + x_T \frac{\partial^2 x_T}{\partial d_T^2} \right]}.$$

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